

The Taylor expansion of Ruelle L-function at the origin and the Borel regulator

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Abstract

We will prove that Ruelle L-function for a cuspidal unitary local system on an odd dimensional complete hyperbolic manifold with finite volume satisfies a functional equation and an analogue of Riemann hypothesis. We will also compute its Laurent expansion at the origin and will prove that the second coefficient coincides with a rational multiple of the volume up to a certain contribution from cusps. Moreover if the dimension is three we will identify the leading coefficient. Both of them will be interpreted as a period of a certain element of K-group of \mathbb{C} . Also a relation with the L^2 -torsion will be discussed.

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1 Introduction

Researches of an L-functions may be roughly classified in the following three subjects:

1. a functional equation,
2. (Riemann hypothesis) a distribution of zeros and poles,
3. an arithmetic or a geometric meaning of special values.

For example let us consider the zeta function for a number field F :

$$\zeta_F(s) = \prod_{\mathfrak{p}} (1 - e^{-s \log N(\mathfrak{p})})^{-1}.$$

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Here \mathfrak{P} runs thorough all prime ideals and $N(\mathfrak{P})$ is the norm. Then $\zeta_F(s)$ satisfies a functional equation and the Riemann hypothesis is still a far reaching problem. It has a zero at $s = 0$ of order $r_1 + r_2 - 1$ where r_1 (resp. r_2) is the number of real (resp. complex) places and the leading coefficient of the Taylor expansion is given by

$$\lim_{s \rightarrow 0} s^{-(r_1+r_2-1)} \zeta_F(s) = -\frac{\#\text{Pic}(\mathcal{O}_F)}{\#(\mathcal{O}_F^\times)_{tors}} \cdot R.$$

R is the covolume of the image of $\mathcal{O}_F^\times \oplus \mathbb{Z}$ by the classical regulator which is defined by the logarithmic function:

$$\mathcal{O}_F^\times \oplus \mathbb{Z} \xrightarrow{r_{1,F}} \mathbb{R}^{r_1+r_2}.$$

The observation that $K_1(\mathcal{O}_F)$ is isomorphic to \mathcal{O}_F^\times and the fact that the order of $\zeta_F(s)$ at $s = 1 - l$ is equal to the dimension d_l of $K_{2l-1}(F) \otimes \mathbb{Q}$ for $l \geq 2$ lead Lichtenbaum to a conjecture; There should be a map

$$K_{2l-1}(F) \xrightarrow{r_{l,F}} \mathbb{R}^{d_l}.$$

such that

$$\lim_{s \rightarrow 1-l} (s+1-l)^{-d_l} \zeta_F(s) = \text{vol}(\mathbb{R}^{d_l} / r_{l,F}(K_{2l-1}(F))).$$

This was solved by Borel. He has also constructed a map

$$K_{2l-1}(\mathbb{C}) \xrightarrow{r_l} \mathbb{R},$$

and an each of $r_{l,F}$ and r_l is referred as *the Borel regulator*. In this paper, under a certain condition, we will show that Ruelle L-function for a unitary local system on an odd dimensional hyperbolic manifold (especially a threefold) with finite volume carries similar properties.

Let X be a complete hyperbolic d -fold ($d = 2n + 1$, $n \geq 1$) of finite volume. Thus it is a quotient of the Poincaré upper half space \mathbb{H}^d by a torsion free discrete subgroup Γ in $\text{SO}_0(d, 1)$, the connected component of the isometry groups $\text{SO}(d, 1)$ of \mathbb{H}^d . Notice that there is the natural bijection between a set of hyperbolic conjugacy classes Γ_{hyp} of Γ and a set of closed geodesics of X . By this the length $l(\gamma)$ of a hyperbolic conjugacy class γ is defined to be one of the corresponding closed geodesic. A closed geodesic will be referred as *prime* if it is not a positive multiple of an another one. Using the bijection we define a subset Γ_{prim} of hyperbolic conjugacy classes which consists of elements corresponding to prime closed geodesics. Let ρ be a unitary representaion of Γ with degree r , i.e. the dimension of the representation space V_ρ is r . Now Ruelle L-function is defined to be

$$R_X(z, \rho) = \prod_{\gamma \in \Gamma_{prim}} \det[1 - \rho(\gamma)e^{-zl(\gamma)}]^{-1}.$$

It absolutely converges if $\text{Re } s > 2n$ and is meromorphically continued to the whole plane ([14], see also §2.4). Hereafter, otherwise mentioned, we will assume that ρ is *cuspidal* (see §2).

We will show that $R_X(z, \rho)$ satisfies a functional equation

$$R_X(z, \rho) \cdot R_X(-z, \rho)^{-1} = \exp\left[\frac{\text{vol}(X)}{\pi} Y(z) + 4 \sum_{j=0}^n (-1)^j \delta(X, \rho) z^j\right],$$

where $Y(z)$ is a polynomial of rational coefficients which vanishes at $z = 0$ and $\delta(X, \rho)$ is a certain constant determined by special values of Epstein L-functions of the fundamental groups at cusps (see §2.3). Notice that here and hereafter if X is closed, since it has no cusp, $\delta(X, \rho)$ does not appear. It will be also shown that its zeros and poles are located on

$$\{z \in \mathbb{C} \mid \text{Re } z = -n, -(n-1), \dots, n-1, n\},$$

except for finitely many of them. For example if $d = 3$, i.e. $n = 1$,

$$R_X(z, \rho) \cdot R_X(-z, \rho)^{-1} = \exp\left[\frac{2r}{\pi} \text{vol}(X) \left(\frac{z^3}{3} - 3z\right)\right].$$

Its logarithmic derivative

$$r_X(z, \rho) + r_X(-z, \rho) = \frac{2r}{\pi} \text{vol}(X) (z^2 - 3), \quad r_X(z, \rho) = \frac{d}{dz} \log R_X(z, \rho),$$

may be compared to the functional equation of Weil conjecture. In fact Hasse-Weil's congruent zeta function of a smooth projective variety M with dimension m over a finite field \mathbb{F}_q is defined to be

$$\zeta_M(z) = \prod_{P \in |M|} (1 - q^{-z \deg(x)})^{-1},$$

where $|M|$ is the set of closed points of M and $\deg(x)$ is the extension degree of the residue field of x over \mathbb{F}_q . Weil conjecture implies that

$$\sigma_M(z) = \frac{d}{dz} \log_q \zeta_M(z + \frac{m}{2}),$$

satisfies

$$\sigma_M(z) + \sigma_M(-z) = \chi(M),$$

where $\chi(M)$ is the Euler characteristic of M . Thus replacing $\chi(M)$ by $2r \text{vol}(X) (z^2 - 3)/\pi$ (which may be not so absurd if we think about Gauss-Bonnet's formula), we find that the logarithmic derivative of Ruelle L-function for a cuspidal unitary local system on a hyperbolic threefold satisfies a functional equation similar to $\sigma_M(s)$.

Let us expand $R_X(z, \rho)$ at the origin which is a symmetric point of the functional equation:

$$R_X(z, \rho) = c_0 z^h (1 + c_1 z + \cdots), \quad c_0 \neq 0.$$

We are interested in the coefficients c_0 and c_1 .

Theorem 1.1. $c_1 - 2 \sum_{j=0}^n (-1)^j \delta(X, \rho)$ is a rational multiple of $\text{vol}(X)/\pi$.

For example if $d = 3$, we will show

$$c_1 = -3r \frac{\text{vol}(X)}{\pi}.$$

Combining with the results of Goncharov([6]) this yields

Corollary 1.1. There is an element $\gamma_X \in K_{2n+1}(\mathbb{C})$ called the Borel element so that $c_1 - 2 \sum_{j=0}^n (-1)^j \delta(X, \rho)$ is a rational multiple of $r_{n+1}(\gamma_X)/\pi$.

In §4 we will recall a construction of γ_X for a closed hyperbolic threefold and will show

$$c_1 = -\frac{3r}{\pi} \cdot r_2(\gamma_X).$$

It is natural to expect that the leading coefficient c_0 has a similar interpretation. In fact it is true at least if $d = 3$.

Theorem 1.2. Let $h^p(X, \rho)$ be the dimension of $H^p(X, \rho)$. Suppose that $d = 3$. Then $R_X(z, \rho)$ has a zero at the origin of order $2h^1(X, \rho)$ and

$$c_0 = (\tau^*(X, \rho) \cdot \text{Per}(X, \rho))^2.$$

Here $\tau^*(X, \rho)$ and $\text{Per}(X, \rho)$ are the modified Franz-Reidemeister torsion and the period of (X, ρ) , respectively. (See §3.4.) (If $h^1(X, \rho)$ is zero $\tau^*(X, \rho)$ is the usual Franz-Reidemeister torsion $\tau(X, \rho)$.) If X is closed the theorem has been already proved by Fried ([5]). In fact he has proved it for a closed odd dimensional hyperbolic manifold.

Suppose $d = 3$ and that $h^1(X, \rho)$ vanishes. Let us fix a triangulation of X and a unitary basis $\mathbf{e} = \{\mathbf{e}_1, \dots, \mathbf{e}_r\}$ of V_ρ . By Poincaré duality we know that all $H^p(X, \rho)$ vanishes for all p and therefore the cochain complex $C^*(X, \rho)$ is acyclic. Then Milnor has constructed an element $\tau(X, \rho, \mathbf{e})$ in $K_1(\mathbb{C}) \simeq \mathbb{C}^\times$ which is referred as the Milnor element and has shown ([12]):

$$\log \tau(X, \rho) = 2\pi r_1(\tau(X, \rho, \mathbf{e})).$$

Summarizing there are rational numbers α and β such that

$$\log R_X(0, \rho) = \alpha \pi r_1(\tau(X, \rho, \mathbf{e})), \quad \frac{d}{dz} \log R_X(z, \rho)|_{z=0} = \frac{\beta}{\pi} \cdot r_2(\gamma_X).$$

Thus replacing the logarithmic derivative by a shift:

$$f = f(z) \rightarrow f^{[k]}(z) = f(z - k),$$

our formula will correspond to Lichtenbaum conjecture.

In [14], Park has obtained

Fact 1.1. *Let X be an odd dimensional complete hyperbolic manifold with finite volume and ρ a unitary local system on X which may not satisfy the cuspidal condition. Then the leading coefficient of the Laurent expansion at the origin is $\exp(-\zeta'_X(0, \rho))$ where $\zeta_X(s, \rho)$ is the spectral zeta function (see §3.4).*

By Hodge theory $H^p(X, \rho)$ is isomorphic to $\text{Ker} \Delta_X^p$, the kernel of Hodge Laplacian Δ_X^p . They are subspaces of $C^p(X, \rho)$ and $L^2(X, \Omega^p(\rho))$, the space of square integrable sections of p -forms twisted by ρ on X , respectively. (Here notice that $H^p(X, \rho)$ is isomorphic to the kernel of combinatoric Laplacian acting on $C^p(X, \rho)$, see §3.4.) Using this two metrics will be defined on the determinant line bundle $\det H^*(X, \rho)$. One is *Franz-Reidemeister metric* which is defined in terms of the combinatoric L^2 -norm $|\cdot|_{l^2, X}$ induced from the natural metric on $C^*(X, \rho)$ and the modified Franz-Reidemeister torsion:

$$\|\cdot\|_{FR} = |\cdot|_{l^2, X} \cdot \tau^*(X, \rho)^{1/2}.$$

The other is *Ray-Singer metric*:

$$\|\cdot\|_{RS} = |\cdot|_{L^2, X} \cdot \exp\left(-\frac{1}{2}\zeta'_X(0, \rho)\right),$$

where $|\cdot|_{L^2, X}$ is the analytic L^2 -norm derived from the inner product on $L^2(X, \Omega^p(\rho))$. Since by definition $\text{Per}(X, \rho)$ is $|\cdot|_{l^2, X} / |\cdot|_{L^2, X}$, **Theorem 2.1** is reduced to show the following Cheeger-Müller type theorem.

Theorem 1.3.

$$\|\cdot\|_{FR} = \|\cdot\|_{RS}.$$

For a convenience we will give a proof of **Fact 1.1** in **Appendix** under an assumption that $d = 3$ and ρ is cuspidal. In particular the last assumption implies that it is not necessary to take care of the scattering term in Selberg trace formula and one can prove the desired result just following Fried's argument ([5]).

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2 The second coefficient

Let

$$\mathbb{H}^d = \{(x_1, \dots, x_{d+1}) \mid x_1^2 + \dots + x_d^2 - x_{d+1}^2 = -1, x_{d+1} > 0\}$$

be the hyperbolic space form ($d = 2n + 1$, $n \geq 1$). We will choose its origin to be $\mathbf{o} = (0, \dots, 0, 1)$. The connected component $G = \mathrm{SO}_o(d, 1)$ of its isometry group $\mathrm{SO}(d, 1)$ transitively acts on \mathbb{H}^d and the isotropy subgroup at \mathbf{o} is a maximal compact subgroup $K = \mathrm{SO}(d)$. Thus we have a surjective map

$$G \xrightarrow{\pi} \mathbb{H}^d, \quad \pi(g) = g \cdot \mathbf{o},$$

which induces a diffeomorphism

$$G/K \simeq \mathbb{H}^d. \quad (1)$$

Let \mathfrak{g} be the Lie algebra of G and θ the Cartan involution. We define *the normalized Cartan-Killing form* to be

$$(X, Y) = -\frac{1}{4\pi} \mathrm{Tr}(adX \circ ad(\theta Y)), \quad X, Y \in \mathfrak{g}.$$

The Cartan involution provides a decomposition of the Lie algebra \mathfrak{g} of G :

$$\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{h},$$

where \mathfrak{k} and \mathfrak{h} are the $+1$ - and -1 -eigenspaces, respectively. \mathfrak{h} may be identified with the tangent space of \mathbb{H}^n at the origin and the normalized Cartan-Killing form defines a Riemannian metric on \mathbb{H}^n with constant curvature -1 . Let \mathfrak{a} be the maximal abelian subalgebra of \mathfrak{h} and β the positive restricted root of $(\mathfrak{g}, \mathfrak{a})$. Then \mathfrak{a} is one dimensional and let us choose $H \in \mathfrak{a}$ satisfying $\beta(H) = 1$. Then the Lie subgroup A of \mathfrak{a} is isomorphic to \mathbb{R} by a map:

$$\mathbb{R} \simeq A, \quad t \mapsto \exp(tH). \quad (2)$$

Let \mathfrak{n} be the positive root space of β and $N = \exp(\mathfrak{n})$ the associated Lie subgroup. Then using the Iwasawa decomposition

$$G = KAN,$$

we introduce a Haar measure on G by

$$dg = a^{2\rho} dk \cdot da \cdot dn.$$

Here $\rho = n\beta$ is the half sum of positive roots of $(\mathfrak{g}, \mathfrak{a})$ and $a^{2\rho} = \exp(2\rho(\log a))$. dk is the Haar measure on K whose total mass is one and da is the push forward of the Lebesgue measure on \mathbb{R} by (2). The volume form dn of N is induced by the normalized Cartan-Killing form. Let $M \simeq \mathrm{SO}(d-1)$ be the centralizer of A in K and $P = MAN$ a proper parabolic subgroup.

Let X be a complete hyperbolic d -fold with finite volume and $\{\infty_1, \dots, \infty_h\}$ be the cusps. Thus it is a quotient of \mathbb{H}^d by a torsion free discrete subgroup Γ in G . A conjugate $P_\nu = g_\nu P g_\nu^{-1}$ ($g_\nu \in G$) corresponds to a cusp ∞_ν and *the fundamental group* Γ_ν at ∞_ν is defined to be

$$\Gamma_\nu = \Gamma \cap P_\nu.$$

We will normalize so that $\infty_1 = \infty$ and g_1 is the identity. Since Γ is torsion free Γ_ν is equal to $\Gamma \cap N_\nu$ ($N_\nu = g_\nu N g_\nu^{-1}$) which is a lattice in \mathbb{R}^{2n} . Let ρ be a unitary representation of Γ of degree r and ρ_ν its restriction to Γ_ν . Since Γ_ν is abelian ρ_ν is decomposed into a direct sum of characters:

$$\rho_\nu = \oplus_{i=1}^r \chi_{\nu,i}.$$

Throughout the paper we will assume that ρ is *cuspidal*, i.e. *none* of $\{\chi_{\nu,i}\}_{1 \leq i \leq r, 1 \leq \nu \leq h}$ is trivial. This terminology will be justified in **Lemma 3.1**.

Let Ω_X^j be the vector bundle of j -forms on X and $\Omega_X^j(\rho)$ its twist by ρ . Then the pullback Ω^j of Ω_X^j on \mathbb{H}^d is a homogeneous vector bundle. In fact let ξ be the standard action of $\mathrm{SO}(d)$ on \mathbb{R}^d and $\mathrm{SO}(d) \xrightarrow{\xi_j} \mathrm{GL}(\wedge^j \mathbb{R}^d)$ its exterior product. Then Ω^j is isomorphic to $\mathrm{SO}(d, 1) \times_{\mathrm{SO}(d), \xi_j} \wedge^j \mathbb{R}^d$. By an inclusion:

$$\mathrm{SO}(d-1) = \mathrm{SO}(2n) \rightarrow \mathrm{SO}(d), \quad A \mapsto \begin{pmatrix} A & 0 \\ 0 & 1 \end{pmatrix}$$

the restriction of ξ to $\mathrm{SO}(2n)$ is decomposed into a direct sum of the standard representation σ of $\mathrm{SO}(2n)$ on \mathbb{R}^{2n} and the trivial module **1**. Therefore we have

$$\xi_j|_{\mathrm{SO}(2n)} \simeq \sigma_j \oplus \sigma_{j-1},$$

where σ_j is the j -th exterior product of σ . Let us observe that σ_0 and σ_{2n} are trivial and σ_j is isomorphic to σ_{2n-j} . $\sigma_j \otimes \mathbb{C}$ is irreducible for $j \neq n$ whereas $\sigma_n \otimes \mathbb{C}$ splits into a direct sum of two irreducible representations, σ_n^+ and σ_n^- . We prepare notation. Let $\gamma \in \Gamma$ be a hyperbolic element. Then it is conjugate an element of MA , $m_\gamma \exp[l(\gamma)H]$ ($m_\gamma \in M$, $\exp[l(\gamma)H] \in A$) where $l(\gamma)$ is the length of the conjugacy class of γ . There is a $\gamma_0 \in \Gamma$ which determines a prime conjugacy class and that $\gamma = \gamma_0^{\mu(\gamma)}$, where $\mu(\gamma)$ is a positive integer. For $0 \leq j \leq 2n$ we put

$$\alpha_j(\gamma) = \frac{\mathrm{Tr} \rho(\gamma) \cdot \mathrm{Tr} \sigma_j(m_\gamma) \cdot l(\gamma_0)}{\Delta(\gamma)}, \quad c_j = |n - j|$$

and

$$S_j(z) = \exp\left[- \sum_{\gamma \in \Gamma_{hyp}} \frac{\alpha_j(\gamma)}{l(\gamma)} e^{-zl(\gamma)}\right],$$

where

$$\Delta(\gamma) = \det(I_{d-1} - e^{-l(\gamma)} m_\gamma).$$

Let s_j be the logarithmic derivative of S_j :

$$s_j(z) = \frac{d}{dz} \log S_j(z) = \sum_{\gamma \in \Gamma_{hyp}} \alpha_j(\gamma) e^{-zl(\gamma)}.$$

If $\mathrm{Re} z$ is sufficiently large $S_j(z)$ absolutely converges.

Fact 2.1. ($[5]$, (RS))

$$R_X(z, \rho) = \prod_{j=0}^{2n} S_j(z+j)^{(-1)^{j+1}}.$$

An isomorphism $\sigma_j \simeq \sigma_{2n-j}$ induces

$$\alpha_j(\gamma) = \alpha_{2n-j}(\gamma) \quad \text{and} \quad S_j(z) = S_{2n-j}(z),$$

and therefore

Lemma 2.1.

$$\begin{aligned} r_X(z, \rho) &= \sum_{j=0}^{2n} (-1)^{j+1} s_j(z+j) \\ &= \sum_{j=0}^{n-1} (-1)^{j+1} \{s_j(z+j) + s_j(z+2n-j)\} + (-1)^{n+1} s_n(z+n). \end{aligned}$$

Lemma 2.2. *Let f be a meromorphic function defined on a neighborhood of the origin and*

$$f(z) = a_0 z^h (1 + a_1 z + \cdots), \quad a_0 \neq 0$$

its Laurent expansion. Then

$$a_1 = \frac{1}{2} \lim_{z \rightarrow 0} \left\{ \frac{f'}{f}(z) + \frac{f'}{f}(-z) \right\}.$$

Proof. Set $f(z) = a_0 z^h g(z)$, where $g(z) = 1 + a_1 z + \cdots$. Then

$$\frac{f'}{f}(z) = \frac{h}{z} + \frac{g'}{g}(z).$$

and therefore

$$\lim_{z \rightarrow 0} \left\{ \frac{f'}{f}(z) + \frac{f'}{f}(-z) \right\} = \frac{2g'(0)}{g(0)} = 2a_1.$$

□

Let us regard a meromorphic continuation of $R_X(z, \rho)$ for a moment and

$$R_X(z, \rho) = c_0 z^h (1 + c_1 z + \cdots), \quad c_0 \neq 0$$

the Taylor expansion at the origin. By **Lemma 2.2** we obtain

$$c_1 = \frac{1}{2} \lim_{z \rightarrow 0} \{r_X(z, \rho) + r_X(-z, \rho)\}.$$

Using Selberg trace formula we will compute RHS. Let Δ_X^j be Hodge Laplacian acting on the space of smooth sections of $\Omega_X^j(\rho)$ and its selfadjoint extension to

$L^2(X, \Omega_X^j(\rho))$ will be denoted by the same character. Since ρ is cuspidal Δ_X^j has only discrete spectrum which do not accumulate and Selberg trace formula for the heat kernel becomes

$$\mathrm{Tr}[e^{-t\Delta_X^j}] = H_j(t) + I_j(t) + U_j(t), \quad t > 0.$$

Here $H_j(t)$, $I_j(t)$ and $U_j(t)$ are the hyperbolic, the identical and the unipotent orbital integral, respectively([17]). In this section we will compute the derivative of Laplace transform of an each of them:

$$L(f)(z) = 2z \int_0^\infty e^{-tz^2} f(t) dt, \quad f = H_j, I_j, U_j.$$

2.1 The hyperbolic orbital integral

For $0 \leq j \leq 2n$, let us put

$$h_j(t) = \frac{1}{\sqrt{4\pi t}} \sum_{\gamma \in \Gamma_{hyp}} \alpha_j(\gamma) \exp\left\{-\left(\frac{l(\gamma)^2}{4t} + tc_j^2 + nl(\gamma)\right)\right\}.$$

Then a hyperbolic orbital integral is given by ([5], **Theorem 2**)

$$H_j(t) = h_j(t) + h_{j-1}(t),$$

where $h_{-1}(t)$ is understood to be 0. Notice that

$$h_j(t) = h_{2n-j}(t), \quad 0 \leq j \leq n$$

Although the following lemma seems to be well known, we will give a proof for a completeness.

Lemma 2.3. *Let l and z be positive numbers. Then*

$$\int_0^\infty \frac{1}{\sqrt{\pi t}} e^{-z^2 t - \frac{t^2}{4t}} dt = \frac{e^{-lz}}{z}.$$

Proof. Let us remember the well known formula:

$$\int_0^\infty e^{-t^2 - \frac{x^2}{t^2}} dt = \frac{\sqrt{\pi}}{2} e^{-2x}, \quad x > 0.$$

If we differentiate it with respect to x , we obtain

$$x \int_0^\infty \frac{1}{t^2} e^{-t^2 - \frac{x^2}{t^2}} dt = \frac{\sqrt{\pi}}{2} e^{-2x}.$$

A change of variables, $t = z\sqrt{y}$, $x = \frac{lz}{2}$ will yield

$$\frac{1}{\sqrt{4\pi}} \int_0^\infty y^{-\frac{3}{2}} e^{-z^2 y - \frac{t^2}{4y}} dy = \frac{e^{-lz}}{l}.$$

Take a derivative of this equation with respect to z , the desired formula will be proved.

□

Therefore if z is a sufficiently large positive number,

$$\begin{aligned}
L(e^{tc_j^2} h_j)(z) &= 2z \int_0^\infty \sum_{\gamma \in \Gamma_{hyp}} \alpha_j(\gamma) e^{-nl(\gamma)} \frac{1}{\sqrt{4\pi t}} e^{-z^2 t - \frac{l(\gamma)^2}{4t}} dt \\
&= \sum_{\gamma \in \Gamma_{hyp}} \alpha_j(\gamma) e^{-nl(\gamma)} z \int_0^\infty \frac{1}{\sqrt{\pi t}} e^{-z^2 t - \frac{l(\gamma)^2}{4t}} dt \\
&= \sum_{\gamma \in \Gamma_{hyp}} \alpha_j(\gamma) e^{-(z+n)l(\gamma)} \\
&= s_j(z+n).
\end{aligned}$$

and we have proved the following proposition.

Proposition 2.1. *For a sufficiently large positive number z ,*

$$L(e^{tc_j^2} h_j)(z) = s_j(z+n). \quad (3)$$

Park has obtained the following proposition even though ρ is not cuspidal ([14]). For the sake of a convenience, we will give a proof in §2.4 under our assumption.

Proposition 2.2. *$s_j(z)$ is continued to the entire plane as a meromorphic function whose singularities are at most only simple poles with integral residues.*

This implies a meromorphic continuation of S_j to the whole plane. Therefore by **Fact 2.1** $R_X(z, \rho)$ is also meromorphically continued.

2.2 The identical orbital integral

We put

$$i_j(t) = i_{2n-j}(t) = \frac{r}{4\pi} \text{vol}(\Gamma \backslash G) \int_{-\infty}^\infty e^{-t(\lambda^2 + c_j^2)} P_j(\lambda) d\lambda, \quad 0 \leq j \leq n-1,$$

and

$$i_n(t) = \frac{r}{2\pi} \text{vol}(\Gamma \backslash G) \int_{-\infty}^\infty e^{-t\lambda^2} P_n(\lambda) d\lambda.$$

Here $\text{vol}(\Gamma \backslash G)$ is the volume of $\Gamma \backslash G$ and P_j is the Plancherel measure for σ_j ([10]):

$$P_j(\lambda) = \frac{4^{1-n}}{(2n-1)!!^2 \pi} \binom{2n}{j} q_j(\lambda),$$

where

$$q_j(\lambda) = \prod_{k=1}^j \{\lambda^2 + (n-k+1)^2\} \prod_{k=j+1}^n \{\lambda^2 + (n-k)^2\}$$

Then the identical orbital integral is given by

$$I_j(t) = i_j(t) + i_{j-1}(t).$$

Since Γ is torsin free its intersection with K is the only identity element. Remember that we have normalized the Haar measure so that $\text{vol}(K)$ to be one and thus

$$\text{vol}(\Gamma \backslash G) = \text{vol}(\Gamma \backslash \mathbb{H}^d) \cdot \text{vol}(K) = \text{vol}(X).$$

Hence we have obtained

$$i_j(t) = i_{2n-j}(t) = \frac{r}{4\pi} \text{vol}(X) \int_{-\infty}^{\infty} e^{-t(\lambda^2 + c_j^2)} P_j(\lambda) d\lambda, \quad 0 \leq j \leq n-1,$$

and

$$i_n(t) = \frac{r}{2\pi} \text{vol}(X) \int_{-\infty}^{\infty} e^{-t\lambda^2} P_n(\lambda) d\lambda.$$

For example if $d = 3$, i.e. $n = 1$, using

$$\int_{-\infty}^{\infty} e^{-t\lambda^2} d\lambda = \sqrt{\pi} t^{-\frac{1}{2}}, \quad (4)$$

one can see

$$i_0(t) = i_2(t) = \frac{r}{4\pi^2} \text{vol}(X) \int_{-\infty}^{\infty} e^{-t(\lambda^2 + 1)} \lambda^2 d\lambda = \frac{r \cdot \text{vol}(X)}{8\pi\sqrt{\pi}} e^{-t} t^{-\frac{3}{2}}, \quad (5)$$

$$i_1(t) = \frac{r}{\pi^2} \text{vol}(X) \int_{-\infty}^{\infty} e^{-t\lambda^2} (\lambda^2 + 1) d\lambda = \frac{r \cdot \text{vol}(X)}{2\pi\sqrt{\pi}} (2t^{-\frac{1}{2}} + t^{-\frac{3}{2}}). \quad (6)$$

In order to compute $L(e^{tc_j^2} i_j)(z)$ let us expand $q_j(\lambda)$ as

$$q_j(\lambda) = \sum_{k=0}^n \gamma_{j,k} \lambda^{2k},$$

where $\gamma_{j,k}$ is an integer and $\gamma_{j,n} = 1$.

Lemma 2.4. *Let z be a positive number. Then for a nonnegative integer k ,*

$$L\left(\int_{-\infty}^{\infty} e^{-t\lambda^2} \lambda^{2k} d\lambda\right)(z) = (-1)^k 2\pi z^{2k}.$$

In particular $L(\int_{-\infty}^{\infty} e^{-t\lambda^2} \lambda^{2k} d\lambda)(z)$ is entirely continued to the whole plane.

Proof. Let us take a k -times derivative of (4) with respect to t . Then we obtain

$$\int_{-\infty}^{\infty} e^{-t\lambda^2} \lambda^{2k} d\lambda = 2^{-k} (2k-1)!! \sqrt{\pi} t^{-\frac{1}{2}-k},$$

and

$$L\left(\int_{-\infty}^{\infty} e^{-t\lambda^2} \lambda^{2k} d\lambda\right)(z) = 2^{-k} (2k-1)!! \sqrt{\pi} L(t^{-\frac{1}{2}-k})(z).$$

Now since

$$\begin{aligned}
L(t^{-\frac{1}{2}-k})(z) &= 2z \int_0^\infty e^{-tz^2} t^{-\frac{1}{2}-k} dt \\
&= 2\Gamma\left(\frac{1}{2} - k\right) z^{2k} \\
&= \frac{(-1)^k 2^{k+1} \sqrt{\pi}}{(2k-1)!!} z^{2k},
\end{aligned}$$

the desired equation has been proved. \square

Proposition 2.3. *We have*

$$L(e^{tc_j^2} i_j)(z) = L(e^{tc_{2n-j}^2} i_{2n-j})(z) = \frac{4^{1-n} r}{2(2n-1)!!^2 \pi} \binom{2n}{j} \text{vol}(X) \sum_{k=0}^n (-1)^k \gamma_{j,k} z^{2k}.$$

for $0 \leq j \leq n-1$ and

$$L(i_n)(z) = \frac{4^{1-n} r}{(2n-1)!!^2 \pi} \binom{2n}{n} \text{vol}(X) \sum_{k=0}^n (-1)^k \gamma_{n,k} z^{2k}.$$

Corollary 2.1. *Suppose $n = 1$. Then*

$$L(e^t i_0)(z) = L(e^t i_2)(z) = -\frac{r}{2\pi} \text{vol}(X) z^2$$

and

$$L(i_1)(z) = \frac{2r}{\pi} \text{vol}(X) (1 - z^2).$$

2.3 The unipotent orbital integral

Let $\zeta_\nu(s, \chi_{\nu,i})$ be the Epstein L-function:

$$\zeta_\nu(s, \chi_{\nu,i}) = \sum_{0 \neq \eta \in \Gamma_\nu} \chi_{\nu,i}(\eta) |X_\eta|^{-2n(s+1)},$$

where X_η is an element of the Lie algebra of N_ν such that $\exp(X_\eta) = \eta$. The norm is taken with respect to the normalized Cartan-Killing form. It absolutely converges if $\text{Re } s$ is sufficiently large and is meromorphically continued to the whole plane. Since $\chi_{\nu,i}$ is nontrivial it is regular at the origin and we put

$$\tau_\nu = \sum_{i=1}^r \zeta_\nu(0, \chi_{\nu,i}).$$

Let $A(\mathfrak{n})$ be the volume of the unit sphere in \mathfrak{n} . By [13] we find the unipotent orbital integral is given by

$$U_j(t) = u_j(t) + u_{j-1}(t),$$

where

$$u_j(t) = u_{2n-j}(t) = \frac{1}{2\pi A(\mathbf{n})} \sum_{\nu=1}^h \text{vol}(\Gamma_\nu \setminus N_\nu) \tau_\nu \int_{-\infty}^{\infty} e^{-t(\lambda^2 + c_j^2)} d\lambda \quad (7)$$

$$= \frac{1}{2\sqrt{\pi} A(\mathbf{n})} \sum_{\nu=1}^h \text{vol}(\Gamma_\nu \setminus N_\nu) \tau_\nu e^{-tc_j^2} t^{-\frac{1}{2}}. \quad (8)$$

for $0 \leq j \leq n-1$ and

$$u_n(t) = \frac{1}{\pi A(\mathbf{n})} \sum_{\nu=1}^h \text{vol}(\Gamma_\nu \setminus N_\nu) \tau_\nu \int_{-\infty}^{\infty} e^{-t\lambda^2} d\lambda \quad (9)$$

$$= \frac{1}{\sqrt{\pi} A(\mathbf{n})} \sum_{\nu=1}^h \text{vol}(\Gamma_\nu \setminus N_\nu) \tau_\nu t^{-\frac{1}{2}}. \quad (10)$$

Here we have used (4).

Lemma 2.5. *Let z be a positive number. Then*

$$L\left(\int_{-\infty}^{\infty} e^{-t\lambda^2} d\lambda\right)(z) = 2\pi,$$

and $L\left(\int_{-\infty}^{\infty} e^{-t\lambda^2} d\lambda\right)(z)$ is entirely continued to the whole plane as 2π .

Proof. We compute,

$$\begin{aligned} L\left(\int_{-\infty}^{\infty} e^{-t\lambda^2} d\lambda\right)(z) &= 2z \int_0^{\infty} dt e^{-tz^2} \int_{-\infty}^{\infty} e^{-t\lambda^2} d\lambda \\ &= 2z \int_{-\infty}^{\infty} d\lambda \int_0^{\infty} e^{-t(\lambda^2 + z^2)} dt \\ &= 2z \int_{-\infty}^{\infty} \frac{d\lambda}{\lambda^2 + z^2}. \end{aligned}$$

The desired formula will be obtained by the contour integration.

□

Thus putting

$$\delta(X, \rho) = \frac{1}{A(\mathbf{n})} \sum_{\nu=1}^h \text{vol}(\Gamma_\nu \setminus N_\nu) \tau_\nu,$$

we have proved the following proposition.

Proposition 2.4. *For $0 \leq j \leq n-1$, both $L(e^{tc_j^2} u_j)(z)$ and $L(e^{tc_{2n-j}^2} u_{2n-j})(z)$ are analytically continued to the entire plane as a constant $\delta(X, \rho)$, whereas $L(u_n)(z)$ is continued as $2\delta(X, \rho)$.*

2.4 An application of Selberg trace formula

Let $0 \leq j \leq n$. Then by definition we know

$$h_j(t) = \sum_{k=0}^j (-1)^{j-k} H_k(t), \quad i_j(t) = \sum_{k=0}^j (-1)^{j-k} I_k(t)$$

and

$$u_j(t) = \sum_{k=0}^j (-1)^{j-k} U_k(t).$$

If we put

$$\delta_j(t) = \sum_{k=0}^j (-1)^{j-k} \text{Tr}[e^{-t\Delta_X^k}],$$

Selberg trace formula implies

$$\delta_j(t) = h_j(t) + i_j(t) + u_j(t). \quad (11)$$

The following lemma will directly follow from the definition of L .

Lemma 2.6.

$$L(e^{tc_j^2} \delta_j)(-z) = -L(e^{tc_j^2} \delta_j)(z).$$

Now we will prove **Proposition 2.2**. Since ρ is cuspidal, Δ_X^k has only discrete spectrum $\{\sigma_k(l)\}_l$ which do not accumulate and are nonnegative. Let us fix $z \in \mathbb{C}$ so that

$$\text{Re} z^2 > n^2.$$

Then

$$\begin{aligned} L(e^{tc_j^2} \delta_j)(z) &= \sum_{k=0}^j (-1)^{j-k} \sum_l 2z \int_0^\infty e^{-t(z^2 - c_j^2 + \sigma_k(l))} dt \\ &= \sum_{k=0}^j (-1)^{j-k} \sum_l \frac{2z}{z^2 - c_j^2 + \sigma_k(l)}, \end{aligned}$$

and

$$L(e^{tc_j^2} \delta_j)(z - c_j) = \sum_{k=0}^j (-1)^{j-k} \sum_l \left\{ \frac{1}{z - c_j + \sqrt{c_j^2 - \sigma_k(l)}} + \frac{1}{z - c_j - \sqrt{c_j^2 - \sigma_k(l)}} \right\}. \quad (12)$$

Thus $L(e^{tc_j^2} \delta_j)(z)$ is meromorphically continued to the whole plane and has only simple poles with integral residues. Using **Proposition 2.1**, **Proposition 2.3** and **Proposition 2.4** Selberg trace formula imply

$$\begin{aligned} s_j(z + n) &= L(e^{tc_j^2} \delta_j)(z) \\ &- \frac{4^{1-n} r(1 + \delta_{j,n})}{2(2n-1)!!^2 \pi} \binom{2n}{j} \text{vol}(X) \sum_{k=0}^n (-1)^k \gamma_{j,k} z^{2k} \\ &- (1 + \delta_{j,n}) \delta(X, \rho), \end{aligned} \quad (13)$$

where $\delta_{j,n}$ is the Kronecker's delta. This proves **Proposition 2.2**.

□

Using **Lemma 2.1** and (13), the above computation shows

$$\begin{aligned}
r_X(z, \rho) &= \sum_{j=0}^{n-1} (-1)^{j+1} \sum_{k=0}^j (-1)^{j-k} \sum_l \left\{ \frac{1}{z - c_j + \sqrt{c_j^2 - \sigma_k(l)}} + \frac{1}{z - c_j - \sqrt{c_j^2 - \sigma_k(l)}} \right\} \\
&+ \sum_{j=0}^{n-1} (-1)^{j+1} \sum_{k=0}^j (-1)^{j-k} \sum_l \left\{ \frac{1}{z + c_j + \sqrt{c_j^2 - \sigma_k(l)}} + \frac{1}{z + c_j - \sqrt{c_j^2 - \sigma_k(l)}} \right\} \\
&+ (-1)^{n+1} \sum_{k=0}^n (-1)^{n-k} \sum_l \left\{ \frac{1}{z + \sigma_k(l)\sqrt{-1}} + \frac{1}{z - \sigma_k(l)\sqrt{-1}} \right\} \\
&+ E(z),
\end{aligned}$$

where $E(z)$ is an entire function. Thus remembering $h^0(X, \rho) = 0$, we have

$$\text{Res}_{z=0} r_X(z, \rho) = 2 \sum_{l=0}^{n-1} (-1)^l (n-l) h^{l+1}(X, \rho).$$

It is obvious that the zeros and poles of $R_X(z, \rho)$ are located on

$$\Xi = \{z \in \mathbb{C} \mid \text{Re } z = -n, -(n-1), \dots, n-1, n\},$$

except for finitely many of them. For example if $d = 3$ (i.e. $n = 1$), we conclude $R_X(z, \rho)$ has a zero at the origin of order $2h^1(X, \rho)$. If the minimum of spectrum of Δ_X^0 is greater than or equal to 1, all zeros and poles are located on Ξ . The following theorem is a consequence of **Lemma 2.1**, **Lemma 2.6** and (13).

Theorem 2.1.

$$r_X(z, \rho) + r_X(-z, \rho) = \frac{4^{1-n} r \cdot \text{vol}(X)}{(2n-1)!!^2 \pi} \chi(z) + 4 \sum_{j=0}^n (-1)^j \delta(X, \rho),$$

where

$$\chi(z) = \sum_{j=0}^n (-1)^j \binom{2n}{j} \sum_{k=0}^n (-1)^k \gamma_{j,k} \{(z+j-n)^{2k} + (z-j+n)^{2k}\}.$$

Corollary 2.2. *Suppose $d = 3$. Then*

$$r_X(z, \rho) + r_X(-z, \rho) = \frac{2r}{\pi} \text{vol}(X)(z^2 - 3).$$

Noting the order of $R_X(z, \rho)$ is even, an easy computation will show

$$\lim_{z \rightarrow 0} R_X(z, \rho) R_X(-z, \rho)^{-1} = 1.$$

By **Theorem 2.1**,

$$\begin{aligned} \frac{d}{dz} \log(R_X(z, \rho) R_X(-z, \rho)^{-1}) &= r_X(z, \rho) + r_X(-z, \rho) \\ &= \frac{4^{1-n} r \cdot \text{vol}(X)}{(2n-1)!!^2 \pi} \chi(z) + 4 \sum_{j=0}^n (-1)^j \delta(X, \rho), \end{aligned}$$

and therefore $R_X(z, \rho)$ satisfies a functional equation:

$$R_X(z, \rho) \cdot R_X(-z, \rho)^{-1} = \exp\left[\frac{4^{1-n} r \cdot \text{vol}(X)}{(2n-1)!!^2 \pi} X(z) + 4 \sum_{j=0}^n (-1)^j \delta(X, \rho) z^j\right],$$

where $X(z)$ is the primitive function of $\chi(z)$ so that $X(0) = 0$. For example if $d = 3$,

$$R_X(z, \rho) \cdot R_X(-z, \rho)^{-1} = \exp\left[\frac{2r}{\pi} \text{vol}(X) \left(\frac{z^3}{3} - 3z\right)\right].$$

Now the second coefficient of the Taylor expansion is obtained by **Lemma 2.2** and **Theorem 2.1**.

Theorem 2.2. *Let*

$$R_X(z, \rho) = c_0 z^h (1 + c_1 z + \cdots), \quad c_0 \neq 0,$$

be the Laurent expansion. Then $c_1 - 2 \sum_{j=0}^n (-1)^j \delta(X, \rho)$ is a rational multiple of $\text{vol}(X)/\pi$.

Corollary 2.3. *Suppose $d = 3$. Then*

$$c_1 = -\frac{3r}{\pi} \text{vol}(X).$$

3 The leading coefficient

Throughout this section we assume that X is a hyperbolic threefold with finite volume. We will compute the leading coefficient of the Taylor expansion of $R_X(z, \rho)$ at the origin. In §2.4 we have seen $\text{ord}_{z=0} R_X(z, \rho) = 2h^1(X, \rho)$. The following fact is a special case of [14].

Fact 3.1.

$$\lim_{z \rightarrow 0} z^{-2h^1(X, \rho)} R_X(z, \rho) = \exp(-\zeta'_X(0, \rho)).$$

Here

$$\zeta_X(z, \rho) = \sum_{p=0}^3 (-1)^p p \cdot \zeta_X^{(p)}(z, \rho),$$

where

$$\zeta_X^{(p)}(z, \rho) = \frac{1}{\Gamma(z)} \int_0^\infty \{\text{Tr}[e^{-t\Delta_X^p}] - h^p(X, \rho)\} t^{z-1} dt.$$

$\zeta_X^{(p)}(z, \rho)$ absolutely converges if $\operatorname{Re} z$ is sufficiently large and is meromorphically continued to the whole plane. In fact let us put

$$\theta_p(t) = \operatorname{Tr}[e^{-t\Delta_X^p}] - h^p(X, \rho).$$

Then the computation of orbital integrals in §2 and Selberg trace formula show that it has an asymptotic expansion on $(0, 1]$ such that

$$\theta_p(t) \sim t^{-\frac{3}{2}} \sum_{l=0}^N c_l t^l + O(t^{N-\frac{3}{2}}). \quad (14)$$

Therefore if $\operatorname{Re} z > N - 3/2$,

$$\int_0^1 \theta_p(t) t^{z-1} dt = \sum_{l=0}^N \frac{c_l}{z + l - \frac{3}{2}} + R_N(z),$$

where $R_N(z)$ is a regular function on $\{z \in \mathbb{C} \mid \operatorname{Re} z > \frac{3}{2} - N\}$ which is meromorphically continued to the whole plane. Since $\theta_p(t)$ exponentially decays as $t \rightarrow \infty$, $\int_1^\infty \theta_p(t) t^{z-1} dt$ is an entire function. Thus writing

$$\int_0^\infty \theta_p(t) t^{z-1} dt = \int_0^1 \theta_p(t) t^{z-1} dt + \int_1^\infty \theta_p(t) t^{z-1} dt,$$

we know that $\zeta_X^{(p)}(z, \rho)$ is meromorphically continued to the whole plane and that it vanishes at the origin. Since we assume that ρ is cuspidal it is possible to prove **Fact 3.1** just following the arguments of [5]. For a convenience we will give a proof in **Appendix**. Thus the leading coefficient is $\exp(-\zeta'_X(0, \rho))$ but we want to express this by a more geometric term.

3.1 Boundary conditions

We will use the Poincaré upper half space model:

$$\mathbb{H}^3 = \{(x, y, r) \in \mathbb{R}^3 \mid r > 0\}, \quad g = \frac{dx^2 + dy^2 + dr^2}{r^2}.$$

For $a \in \mathbb{R}$ we put

$$\mathbb{H}_a^3 = \cap_{\nu=1}^h g_\nu \mathbb{H}_{a,\infty}^3, \quad \mathbb{H}_{a,\infty}^3 = \{(x, y, r) \in \mathbb{H}^3 \mid r \leq e^a\}.$$

(Remember that $g_\nu \in \operatorname{PSL}_2(\mathbb{C})$ is chosen to satisfy

$$N_\nu = g_\nu N g_\nu^{-1}, \quad N = \left\{ \begin{pmatrix} 1 & z \\ 0 & 1 \end{pmatrix} \mid z \in \mathbb{C} \right\}.)$$

Let X_a be the image of \mathbb{H}_a^3 by the natural projection

$$\mathbb{H}^3 \xrightarrow{\pi} X,$$

and Y_a the closure of $X \setminus X_a$. If a is sufficiently large Y_a is a disjoint union of $Y_{a,\nu}$ ($1 \leq \nu \leq h$) which is a warped product of a flat 2-torus $T_\nu = N_\nu/\Gamma_\nu (= \mathbb{R}^2/\Gamma_\nu)$ and an interval $[e^a, \infty)$ with the metric,

$$g = du^2 + e^{-2u}(dx^2 + dy^2).$$

(Here we have made a change of variables: $r = e^u$.) According to the decomposition

$$\rho_\nu = \oplus_{i=1}^r \chi_{\nu,i},$$

a smooth section φ of $\Omega_X^p(\rho)$ over $Y_{a,\nu}$ is written as

$$\varphi = \sum_{i=1}^r \varphi_i, \quad \varphi_i = \sum_{|\alpha|=p} \varphi_{i,\alpha} dx^\alpha \in C^\infty(Y_{a,\nu}, \Omega_X^p(\chi_{\nu,i})),$$

where we have put

$$x_0 = u, \quad x_1 = x, \quad x_2 = y.$$

Lemma 3.1.

$$\int_{T_\nu} \varphi_{i,\alpha} dx dy = 0.$$

Proof. Let us choose $\gamma \in \Gamma_\nu$ so that

$$\chi_{\nu,i}(\gamma) \neq 1.$$

By definition we have

$$\gamma^* \varphi_{i,\alpha} = \chi_{\nu,i}(\gamma) \varphi_{i,\alpha}.$$

and the desired result will follow from

$$\int_{T_\nu} \varphi_{i,\alpha} dx dy = \int_{T_\nu} \gamma^* \varphi_{i,\alpha} dx dy = \chi_{\nu,i}(\gamma) \int_{T_\nu} \varphi_{i,\alpha} dx dy.$$

□

We will consider an eigenvalue problem of Hodge Laplacian Δ_X^p on spaces $L^2(X_a, \Omega_X^p(\rho))$ or $L^2(Y_{a,\nu}, \Omega_X^p(\rho))$ under a certain boundary condition. Hereafter for simplicity we omit the subscript X of Δ_X^p . The restriction $\Omega_X^p(\rho)$ to the boundary T_ν of $Y_{a,\nu}$ is decomposed into

$$\Omega_X^p(\rho)|_{T_\nu} = \Omega_{T_\nu}^p(\rho) \oplus du \wedge \Omega_{T_\nu}^{p-1}(\rho),$$

and according to this a section ω of $\Omega_X^p(\rho)|_{T_\nu}$ is expressed by

$$\omega = \omega_{tan} + \omega_{norm},$$

where ω_{tan} (resp. ω_{norm}) is a section of $\Omega_{T_\nu}^p(\rho)$ (resp. $du \wedge \Omega_{T_\nu}^{p-1}(\rho)$)

Definition 3.1. Let ω be a smooth section of $\Omega_X^p(\rho)$ on X_a or $Y_{a,\nu}$. We call it satisfies the absolute boundary condition if both ω_{norm} and $(d\omega)_{norm}$ vanish on the every boundary. If the Hodge dual $*\omega$ satisfies the absolute boundary condition we will refer that it satisfies the relative boundary condition. If ω satisfies both of the absolute and the relative boundary condition, we call it satisfies the Dirichlet boundary condition.

It is easy to see that ω satisfies the relative boundary condition if and only if both ω_{tan} and $(d\omega)_{tan}$ vanish on every T_ν . Thus ω satisfies the Dirichlet boundary condition if and only if the restrictions of both ω and $d\omega$ become the 0-section of $\Omega_X^p(\rho)|_{T_\nu}$ and $\Omega_X^{p+1}(\rho)|_{T_\nu}$ for every $1 \leq \nu \leq h$. More concretely the latter condition means that if we write

$$\omega = \sum_{|\alpha|=p} f_\alpha dx^\alpha, \quad d\omega = \sum_{|\beta|=p+1} g_\beta dx^\beta,$$

all f_α and g_β vanish along T_ν for every ν . Notice that $*$ interchanges the absolute and relative boundary conditions and preserves the Dirichlet one. Since ρ is unitary the associated local system possesses a fiberwise hermitian inner product Tr_ρ . For $\omega, \eta \in \Omega_X^p(\rho)$ we put

$$(\omega, \eta) = \frac{\text{Tr}_\rho(\omega \wedge *\eta)}{dv_g},$$

which defines a hermitian inner product on $\Omega_X^p(\rho)$. Here dv_g is the volume form of g . Let M be X_a or $Y_{a,\nu}$ and ∇ the covariant derivative. If both ω and η satisfy one of the boundary conditions,

$$\int_M (\Delta^p \omega, \eta) dv_g = \int_M (\nabla \omega, \nabla \eta) dv_g = \int_M (\omega, \Delta^p \eta) dv_g,$$

by Stokes theorem. Therefore Δ^p has a selfadjoint extension $\Delta_{abs}^p, \Delta_{rel}^p$ or Δ_{dir}^p according to a boundary condition. If $*$ is *abs* (resp. *rel* or *dir*) its dual $\hat{*}$ is defined to be *rel* (resp. *abs* or *dir*). Since the Hodge $*$ -operator intertwines the action of Δ_*^p on $L^2(M, \Omega_X^p(\rho))$ and one of $\Delta_{\hat{*}}^{3-p}$ on $L^2(M, \Omega_X^{3-p}(\rho))$, we will only consider the case of $p = 0$ or 1 .

For a later purpose we will introduce one more boundary condition. Let α be a real number greater than one. For a sufficiently large a , $Y_{a,\nu} \cap X_{\alpha a}$ is diffeomorphic to $T_\nu \times [e^a, \alpha e^a]$. If $\omega \in C^\infty(Y_{a,\nu} \cap X_{\alpha a}, \Omega_X^p(\rho))$ satisfies the Dirichlet condition on $T_\nu \times \{e^a\}$ and a condition $*$ ($*$ = *abs*, *rel* or *dir*) on $T_\nu \times \{\alpha e^a\}$ we call it satisfies *Dirichlet/*-condition*. If $\omega \in C^\infty(Y_a \cap X_{\alpha a}, \Omega_X^p(\rho))$ satisfies Dirichlet/*-condition on every connected component ($*$ does not depend on a component) we will refer that it satisfies Dirichlet/*-condition.

3.2 Spectrum of Hodge Laplacian at cusps

Since Δ^p commutes with the action of Γ it preserves the decomposition,

$$C^\infty(Y_{a,\nu}, \Omega_X^p(\rho)) = \oplus_{i=1}^r C^\infty(Y_{a,\nu}, \Omega_X^p(\chi_{\nu,i})).$$

Thus the spectral problem of Hodge Laplacian on $L^2(Y_{a,\nu}, \Omega_X^p(\rho))$ is reduced to one on $L^2(Y_{a,\nu}, \Omega_X^p(\chi_{\nu,i}))$. We will give an explicit formula of Δ^p on $Y_{a,\nu}$. A straightforward computation will show the following lemma.

Lemma 3.2. *Let Δ_T be the positive Laplacian on a flat torus,*

$$\Delta_T = -(\partial_x^2 + \partial_y^2).$$

1. *For $f \in C^\infty(Y_{a,\nu}, \Omega_X^0(\chi_{\nu,i}))$,*

$$\Delta^0 f = e^{2u} \Delta_T f - \partial_u^2 f + 2\partial_u f.$$

2. *For $\omega = f dx + g dy + h du \in C^\infty(Y_{a,\nu}, \Omega_X^1(\chi_{\nu,i}))$,*

$$\begin{aligned} \Delta^1 \omega &= (e^{2u} \Delta_T f - \partial_u^2 f + 2\partial_u f) dx \\ &+ (e^{2u} \Delta_T g - \partial_u^2 g + 2\partial_u g) dy \\ &+ (e^{2u} \Delta_T h - \partial_u^2 h + 2\partial_u h - 2e^{2u}(\partial_x f + \partial_y g)) du. \end{aligned}$$

Fact 3.2. ([16]/**Theorem XIII.1**, *The min-max principle*) *Let A be a selfadjoint operator with domain $D(A)$, which is bounded below. Define*

$$\mu_n(A) = \sup_{\varphi_1, \dots, \varphi_{n-1}} U_A(\varphi_1, \dots, \varphi_{n-1}),$$

where

$$U_A(\varphi_1, \dots, \varphi_{n-1}) = \inf_{\psi \in D(A), \|\psi\|=1, \psi \in \langle \varphi_1, \dots, \varphi_{n-1} \rangle^\perp} (\psi, A\psi),$$

and $\langle \varphi_1, \dots, \varphi_{n-1} \rangle^\perp$ is the orthogonal complement of a vector space $\langle \varphi_1, \dots, \varphi_{n-1} \rangle$ spanned by $\{\varphi_1, \dots, \varphi_{n-1}\}$. Then either of the followings holds:

1. *there are n eigenvalues below the bottom of the essential spectrum and $\mu_n(A)$ is the n -th eigenvalue counting with multiplicity,*
2. *$\mu_n(A)$ is the bottom of the essential spectrum.*

Later on we will need a variant of this.

Lemma 3.3. *Let A be a selfadjoint operator bounded below such that $(A - \lambda)^{-1}$ is compact for a certain $\lambda \in \rho(A)$, where $\rho(A)$ is the resolvent set. Then the n -th eigenvalue $\mu_n(A)$ is obtained by*

$$\mu_n(A) = \inf_{\mathfrak{M} \in \text{Gr}_n D(A)} \sup_{0 \neq v \in \mathfrak{M}} \frac{(Av, v)}{\|v\|^2}.$$

Here $\text{Gr}_n D(A)$ is the set of n -dimensional subspaces of $D(A)$.

Proof. Let $\mu'_n(A)$ be the RHS of the above equation. By the assumption there is a complete orthonormal basis $\{\varphi_n\}_n$ in $D(A)$ such that $A\varphi_n = \mu_n(A)\varphi_n$ with $\mu_1(A) \leq \mu_2(A) \leq \dots$ and $\mu_n(A) \rightarrow \infty$. Let \mathfrak{N} be an n -dimensional space spanned by $\{\varphi_1, \dots, \varphi_n\}$. Thus

$$\mu_n(A) = \sup_{0 \neq v \in \mathfrak{N}} \frac{(Av, v)}{\|v\|^2},$$

and $\mu'_n(A) \leq \mu_n(A)$ by definition. Suppose $\mu'_n(A)$ is strictly less than $\mu_n(A)$. Then there is an n -dimensional subspace \mathfrak{M} of $D(A)$ so that

$$\mu'_n(A) \leq \sup_{0 \neq v \in \mathfrak{M}} \frac{(Av, v)}{\|v\|^2} < \mu_n(A).$$

But by the equation (2a) in pp.77 of [16], the dimension of \mathfrak{M} should be less than n , which is a contradiction. □

Let A and B be selfadjoint operators bounded below which act on a Hilbert space H . Suppose that they have the same domain D and that $A \geq B$, i.e. $(Av, v) \geq (Bv, v)$ for any $v \in D$. Then **Fact 3.1** implies

Lemma 3.4.

$$\mu_n(A) \geq \mu_n(B)$$

Let a and a' be positive numbers so that $a' \geq a$. Extending as 0-map on the outside $L^2(X_a, \Omega_X^p(\rho))$ is embedded into $L^2(X_{a'}, \Omega_X^p(\rho))$. Thus $D(\Delta_{dir}^p|_{X_a})$ is a subspace of $D(\Delta_{dir}^p|_{X_{a'}})$. In particular $\text{Gr}_n(D(\Delta_{dir}^p|_{X_a}))$ is a subset of $\text{Gr}_n(D(\Delta_{dir}^p|_{X_{a'}}))$. Since $\Delta_{dir}^p|_{X_{a'}}$ satisfies the assumption of **Lemma 3.3**,

$$\mu_n(\Delta_{dir}^p|_{X_{a'}}) \leq \mu_n(\Delta_{dir}^p|_{X_a}).$$

Since ρ is cuspidal Δ^p also satisfies the assumption of **Lemma 3.3**. The same argument will yield the following lemma.

Lemma 3.5. 1. Let a and a' be positive numbers so that $a' \geq a$. Then,

$$\mu_n(\Delta_{dir}^p|_{X_{a'}}) \leq \mu_n(\Delta_{dir}^p|_{X_a}).$$

2. For a positive a ,

$$\mu_n(\Delta^p) \leq \mu_n(\Delta_{dir}^p|_{X_a}).$$

and

$$\mu_n(\Delta_*^p|_{X_a}) \leq \mu_n(\Delta_{dir}^p|_{X_a}),$$

where $*$ is abs or rel.

Remark 3.1. The above lemma also follows from the Rayleigh-Ritz technique. ([16] **Theorem XIII.3**)

Let Γ_ν^* be the dual lattice of Γ_ν . We will define its *norm* to be

$$\|\Gamma_\nu^*\| = \text{Min}\{|\gamma| \mid 0 \neq \gamma \in \Gamma_\nu^*\}.$$

Here the modulus $|\cdot|$ is taken with respect to the standard Euclidean metric $dx^2 + dy^2$ on \mathbb{R}^2 .

Proposition 3.1.

$$\mu_1(\Delta_{dir}^0|_{Y_{a,\nu}}) \geq e^{2a}\|\Gamma_\nu^*\|^2.$$

Proof. Let us consider a nonnegative selfadjoint operator

$$P_a = e^{2a}\Delta_T - \partial_u^2 + 2\partial_u$$

on $L^2(Y_{a,\nu}, \Omega^0(\chi_{\nu,i}))$ under Dirichlet condition at the boundary. Since

$$\Delta^0 - P_a = (e^{2u} - e^{2a})\Delta_T$$

is a nonnegative operator **Lemma 3.4** implies

$$\mu_1(\Delta_{dir}^0|_{Y_{a,\nu}}) \geq \mu_1(P_a).$$

For $f \in C_c^\infty(Y_{a,\nu}, \Omega^0(\chi_{\nu,i}))$,

$$\begin{aligned} \int_{Y_{a,\nu}} (P_a f, f) dv_g &= e^{2a} \int_{Y_{a,\nu}} \Delta_T f \cdot \bar{f} e^{-2u} dx dy du + \int_{Y_{a,\nu}} |\partial_u f|^2 e^{-2u} dx dy du \\ &\geq e^{2a} \int_{Y_{a,\nu}} \Delta_T f \cdot \bar{f} e^{-2u} dx dy du \\ &= e^{2a} \int_a^\infty du e^{-2u} \int_{T_\nu} \Delta_T f \cdot \bar{f} dx dy. \end{aligned}$$

Let

$$f = \sum_{\gamma \in \Gamma_\nu^*} \{f_\gamma(u) \mathbf{e}_\gamma(z) + f_\gamma^*(u) \mathbf{e}_\gamma(\bar{z})\}, \quad \mathbf{e}_\gamma(z) = \exp(2\pi i \gamma z)$$

be a Fourier expansion with respect to T_ν -direction. Here notice that by **Lemma 3.1** γ runs through nonzero elements of Γ_ν^* . Then

$$\begin{aligned} \int_{T_\nu} \Delta_T f \cdot \bar{f} &= \text{vol}(T_\nu) \sum_{0 \neq \gamma \in \Gamma_\nu^*} |\gamma|^2 \{|f_\gamma(u)|^2 + |f_\gamma^*(u)|^2\} \\ &\geq \|\Gamma_\nu^*\|^2 \text{vol}(T_\nu) \sum_{0 \neq \gamma \in \Gamma_\nu^*} \{|f_\gamma(u)|^2 + |f_\gamma^*(u)|^2\} \\ &= \|\Gamma_\nu^*\|^2 \int_{T_\nu} |f|^2 dx dy, \end{aligned}$$

and therefore we have obtained

$$\int_{Y_{a,\nu}} (P_a f, f) dv_g \geq e^{2a} \|\Gamma_\nu^*\|^2 \int_{Y_{a,\nu}} (f, f) dv_g.$$

Now **Fact 3.3** implies $\mu_1(P_a) \geq e^{2a} \|\Gamma_\nu^*\|^2$.

□

The same argument will prove

Proposition 3.2. *For $\alpha > 1$ and $*$ = abs or rel,*

$$\mu_1(\Delta_{dir/*}^0|_{X_{\alpha a} \cap Y_{a,\nu}}) \geq e^{2a} \|\Gamma_\nu^*\|^2.$$

Next we will estimate $\mu_1(\Delta_{dir}^1|_{Y_{a,\nu}})$ from below. Before doing this we will give some remarks. Let us fix a positive number α less than a and we make a change of variables,

$$u = v + \alpha.$$

Then $Y_{a,\nu}$ is isometric to a warped product,

$$[a', \infty) \times T'_\nu, \quad a' = a - \alpha,$$

with metric

$$dg = dv^2 + e^{-2v}(dx^2 + dy^2).$$

Here the boundary T'_ν is a quotient of \mathbb{R}^2 with the standard Euclidean metric $dx^2 + dy^2$ by a lattice $e^{-\alpha}\Gamma_\nu$. Thus replacing Γ_ν (resp. T_ν) by $e^{-\alpha}\Gamma_\nu$ (resp. T'_ν) for a sufficiently large α , we may initially assume that $\|\Gamma_\nu^*\| < 1$, or equivalently $\|\Gamma_\nu^*\| > 1$. Taking a sufficiently large we also assume that $e^{2a} > 32$. Let $\omega = fdx + gdy + hdu$ be an element of $C_c^\infty(Y_{a,\nu}, \Omega^1(\chi_{\nu,i}))$. Then a computation in **Proposition 3.1** implies

$$\int_{Y_{a,\nu}} \Delta_T f \cdot \bar{f} dx dy du \geq \|\Gamma_\nu^*\|^2 \int_{Y_{a,\nu}} |f|^2 dx dy du \geq \int_{Y_{a,\nu}} |f|^2 dx dy du, \quad (15)$$

$$\int_{Y_{a,\nu}} \Delta_T g \cdot \bar{g} dx dy du \geq \|\Gamma_\nu^*\|^2 \int_{Y_{a,\nu}} |g|^2 dx dy du \geq \int_{Y_{a,\nu}} |g|^2 dx dy du, \quad (16)$$

and

$$\int_{Y_{a,\nu}} \Delta_T h \cdot \bar{h} e^{-2u} dx dy du \geq \|\Gamma_\nu^*\|^2 \int_{Y_{a,\nu}} |h|^2 e^{-2u} dx dy du. \quad (17)$$

Using

$$\|dx\| = \|dy\| = e^u, \quad \|du\| = 1$$

and **Lemma 3.2**, an integration by parts shows

$$\begin{aligned} \int_{Y_{a,\nu}} (\Delta^1 \omega, \omega) dv_g &= \int_{Y_{a,\nu}} e^{2u} (\Delta_T f \cdot \bar{f} + \Delta_T g \cdot \bar{g}) dx dy du \\ &+ \int_{Y_{a,\nu}} |\nabla_T h|^2 dx dy du \\ &+ \int_{Y_{a,\nu}} (|\partial_u f|^2 + |\partial_u g|^2 + |\partial_u h|^2 e^{-2u}) dx dy du \\ &+ 2 \int_{Y_{a,\nu}} \{(\partial_x h \cdot \bar{f} + \partial_x \bar{h} \cdot f) + (\partial_y h \cdot \bar{g} + \partial_y \bar{h} \cdot g)\} dx dy du \end{aligned}$$

$$= \int_{Y_{a,\nu}} (e^{2u} - 16)(\Delta_T f \cdot \bar{f} + \Delta_T g \cdot \bar{g}) dx dy du \quad (18)$$

$$+ 16 \int_{Y_{a,\nu}} \{(\Delta_T f \cdot \bar{f} - |f|^2) + (\Delta_T g \cdot \bar{g} - |g|^2)\} dx dy du \quad (19)$$

$$+ \frac{1}{4} \int_{Y_{a,\nu}} \{64|f|^2 + 8(\partial_x h \cdot \bar{f} + \partial_x \bar{h} \cdot f) + |\nabla_T h|^2\} dx dy du \quad (20)$$

$$+ \frac{1}{4} \int_{Y_{a,\nu}} \{64|g|^2 + 8(\partial_y h \cdot \bar{g} + \partial_y \bar{h} \cdot g) + |\nabla_T h|^2\} dx dy du \quad (21)$$

$$+ \frac{1}{2} \int_{Y_{a,\nu}} |\nabla_T h|^2 dx dy du \quad (22)$$

$$+ \int_{Y_{a,\nu}} (|\partial_u f|^2 + |\partial_u g|^2 + |\partial_u h|^2 e^{-2u}) dx dy du. \quad (23)$$

By (15) and (16), (19) is nonnegative. Moreover

$$(8|f| - |\nabla_T h|)^2 \leq 64|f|^2 + 8(\partial_x h \cdot \bar{f} + \partial_x \bar{h} \cdot f) + |\nabla_T h|^2$$

and

$$(8|g| - |\nabla_T h|)^2 \leq 64|g|^2 + 8(\partial_y h \cdot \bar{g} + \partial_y \bar{h} \cdot g) + |\nabla_T h|^2$$

imply that both (20) and (21) are nonnegative. Since

$$\begin{aligned} \int_{Y_{a,\nu}} (e^{2u} - 16)(\Delta_T f \cdot \bar{f} + \Delta_T g \cdot \bar{g}) dx dy du &= \int_{Y_{a,\nu}} (e^{2u} - 16)(|\nabla_T f|^2 + |\nabla_T g|^2) dx dy du \\ &\geq (e^{2a} - 16) \int_{Y_{a,\nu}} (|\nabla_T f|^2 + |\nabla_T g|^2) dx dy du \\ &= (e^{2a} - 16) \int_{Y_{a,\nu}} (\Delta_T f \cdot \bar{f} + \Delta_T g \cdot \bar{g}) dx dy du \end{aligned}$$

we obtain

$$\begin{aligned} \int_{Y_{a,\nu}} (\Delta^1 \omega, \omega) dv_g &\geq (e^{2a} - 16) \int_{Y_{a,\nu}} (\Delta_T f \cdot \bar{f} + \Delta_T g \cdot \bar{g}) dx dy du \\ &\quad + \frac{1}{2} e^{2a} \int_{Y_{a,\nu}} |\nabla_T h|^2 e^{-2u} dx dy du \\ &\geq \frac{1}{2} e^{2a} \int_{Y_{a,\nu}} (\Delta_T f \cdot \bar{f} + \Delta_T g \cdot \bar{g} + \Delta_T h \cdot \bar{h} e^{-2u}) dx dy du. \end{aligned}$$

Here we have used the fact e^{2a} is greater than 32. Thus by (15), (16) and (17) we see

$$\int_{Y_{a,\nu}} (\Delta^1 \omega, \omega) dv_g \geq \frac{1}{2} e^{2a} \|\Gamma_\nu^*\|^2 \int_{Y_{a,\nu}} \|\omega\|^2 dv_g.$$

Now **Fact 3.1** yields

Proposition 3.3. *For a sufficiently large a ,*

$$\mu_1(\Delta_{dir}^1|_{Y_{a,\nu}}) \geq \frac{1}{2}e^{2a}||\Gamma_\nu^*||^2.$$

By the same argument we will find

Proposition 3.4. *Suppose $\alpha > 1$. Then for a sufficiently large a and $*$ = abs or rel,*

$$\mu_1(\Delta_{dir/*}^1|_{X_{\alpha a} \cap Y_{a,\nu}}) \geq \frac{1}{2}e^{2a}||\Gamma_\nu^*||^2.$$

3.3 A convergence of spectrum

In **Lemma 3.5** we have shown that $\mu_n(\Delta_{dir}^p|_{X_a})$ is a monotone decreasing function of a which is bounded below by $\mu_n(\Delta^p)$. In this section we will show the following theorems.

Theorem 3.1.

$$\lim_{a \rightarrow \infty} \mu_n(\Delta_{dir}^p|_{X_a}) = \mu_n(\Delta^p).$$

Theorem 3.2.

$$\lim_{a \rightarrow \infty} \mu_n(\Delta_*^p|_{X_a}) = \mu_n(\Delta^p) \quad * = abs, rel.$$

Corollary 3.1. *Let t be a positive number. Then*

$$\text{Tr}[e^{-t\Delta^p}] = \lim_{a \rightarrow \infty} \text{Tr}[e^{-t\Delta_{dir}^p|_{X_a}}] = \lim_{a \rightarrow \infty} \text{Tr}[e^{-t\Delta_*^p|_{X_a}}], \quad * = abs, rel.$$

Let us fix a positive a_0 so that Y_{a_0} is a disjoint union:

$$Y_{a_0} = \coprod_{\nu=1}^h T_\nu \times [a_0, \infty).$$

We may assume that $e^{2a_0} > 32$ and $||\Gamma_\nu^*|| > 1$ for every ν . Thus **Proposition 3.1** and **Proposition 3.3** are available for $a > a_0$. Let us fix such an a and let χ be a smooth function on X satisfying

1. $0 \leq \chi \leq 1$.
2. $\chi|_{X_a} = 1$ and $\chi|_{Y_{2a}} = 0$.
3. $|\nabla \chi| \leq a^{-1}$.

Let φ_i be its eigenform of Δ^p whose eigenvalue is $\mu_i(\Delta^p)$ and \mathfrak{M}_n an element of $\text{Gr}_n D(\Delta^p)$ spanned by $\{\varphi_1, \dots, \varphi_n\}$. Then for an arbitrary $\varphi \in \mathfrak{M}_n$ we have

$$\int_X ||\nabla \varphi||^2 dv_g = \int_X (\Delta^p \varphi, \varphi) dv_g \leq \mu_n(\Delta^p) \int_X ||\varphi||^2 dv_g. \quad (24)$$

The LHS is

$$\begin{aligned}
\int_X \|\nabla \varphi\|^2 dv_g &= \int_X \|\nabla(\chi\varphi) + \nabla((1-\chi)\varphi)\|^2 dv_g \\
&= \int_X \|\nabla(\chi\varphi)\|^2 dv_g + \int_X \|\nabla((1-\chi)\varphi)\|^2 dv_g \\
&+ 2\operatorname{Re} \int_X (\nabla(\chi\varphi), \nabla((1-\chi)\varphi)) dv_g.
\end{aligned}$$

Since

$$\chi(1-\chi) \leq \frac{1}{4} \quad \text{and} \quad |\nabla \chi| \leq \frac{1}{a}$$

and by Schwartz inequality,

$$|(\nabla(\chi\varphi), \nabla((1-\chi)\varphi))| \leq \left(\frac{1}{a} + \frac{1}{a^2}\right) \|\varphi\|^2 + \left(\frac{1}{a} + \frac{1}{4}\right) \|\nabla \varphi\|^2.$$

Therefore (24) implies

$$\begin{aligned}
\mu_n(\Delta^p) \int_X \|\varphi\|^2 dv_g &\geq \int_X \|\nabla((1-\chi)\varphi)\|^2 dv_g + \int_X \|\nabla(\chi\varphi)\|^2 dv_g \\
&- 2\left(\frac{1}{a} + \frac{1}{a^2}\right) \int_X \|\varphi\|^2 dv_g - 2\left(\frac{1}{a} + \frac{1}{4}\right) \int_X \|\nabla \varphi\|^2 dv_g \\
&\geq \int_X \|\nabla((1-\chi)\varphi)\|^2 dv_g \\
&- 2\left\{\frac{1}{a} + \frac{1}{a^2} + \mu_n(\Delta^p)\left(\frac{1}{a} + \frac{1}{4}\right)\right\} \int_X \|\varphi\|^2 dv_g.
\end{aligned}$$

Notice that $(1-\chi)\varphi$ is contained in the domain of $\Delta_{dir}^p|_{Y_a}$. By **Fact 3.2** and **Proposition 3.1** (if $p=0$), or **Proposition 3.3** (if $p=1$),

$$\begin{aligned}
\int_X \|\nabla((1-\chi)\varphi)\|^2 dv_g &\geq \mu_1(\Delta_{dir}^p|_{Y_a}) \int_X \|(1-\chi)\varphi\|^2 dv_g \\
&\geq \mu_1(\Delta_{dir}^p|_{Y_a}) \int_{Y_{2a}} \|\varphi\|^2 dv_g \\
&\geq Ce^{2a} \int_{Y_{2a}} \|\varphi\|^2 dv_g,
\end{aligned}$$

where C is a positive constant independent of a . So we have obtained

$$\{\mu_n(\Delta^p) + 2\left(\frac{1}{a} + \frac{1}{a^2} + \mu_n(\Delta^p)\left(\frac{1}{a} + \frac{1}{4}\right)\right\} \int_X \|\varphi\|^2 dv_g \geq Ce^{2a} \int_{Y_{2a}} \|\varphi\|^2 dv_g.$$

Now put

$$\rho_n(a) = 2C^{-1}e^{-2a} \left\{ \left(\frac{1}{a} + \frac{1}{a^2}\right) + \mu_n(\Delta^p)\left(\frac{1}{a} + \frac{3}{4}\right) \right\},$$

then we have proved the following proposition.

Proposition 3.5. For $\varphi \in \mathfrak{M}_n$

$$\int_{Y_{2a}} \|\varphi\|^2 dv_g \leq \rho_n(a) \int_X \|\varphi\|^2 dv_g.$$

Let α be greater than two and ϕ_i an eigenform of $\Delta_*^p|_{X_{\alpha a}}$ ($*$ = *rel*, or *abs*) whose eigenvalue is $\mu_i(\Delta_*^p|_{X_{\alpha a}})$ and $\mathfrak{M}_n(\alpha a)$ an element of $\text{Gr}(\Delta_*^p|_{X_{\alpha a}})$ spanned by $\{\phi_1, \dots, \phi_n\}$.

Proposition 3.6. For $\phi \in \mathfrak{M}_n(\alpha a)$,

$$\int_{Y_{2a} \cap X_{\alpha a}} \|\phi\|^2 dv_g \leq \rho_n^0(a) \int_{X_{\alpha a}} \|\phi\|^2 dv_g,$$

where

$$\rho_n^0(a) = 2C^{-1}e^{-2a} \left\{ \left(\frac{1}{a} + \frac{1}{a^2} \right) + \mu_n(\Delta_{dir}^p|_{X_{a_0}}) \left(\frac{1}{a} + \frac{3}{4} \right) \right\}.$$

Proof. The argument is almost same as one of **Proposition 3.5**. For $\phi \in \mathfrak{M}_n(\alpha a)$,

$$\int_{X_{\alpha a}} \|\nabla \phi\|^2 dv_g = \int_{X_{\alpha a}} (\Delta^p \phi, \phi) dv_g \leq \mu_n(\Delta_*^p|_{X_{\alpha a}}) \int_{X_{\alpha a}} \|\phi\|^2 dv_g. \quad (25)$$

Using **Proposition 3.2** and **Proposition 3.4** instead **Proposition 3.1** and **Proposition 3.3**, respectively the previous computation will show

$$\begin{aligned} \mu_n(\Delta_*^p|_{X_{\alpha a}}) \int_{X_{\alpha a}} \|\phi\|^2 dv_g &\geq \int_{X_{\alpha a}} \|\nabla \phi\|^2 dv_g \\ &\geq Ce^{2a} \int_{Y_{2a} \cap X_{\alpha a}} \|\phi\|^2 dv_g \\ &\quad - 2 \left\{ \frac{1}{a} + \frac{1}{a^2} + \mu_n(\Delta_*^p|_{X_{\alpha a}}) \left(\frac{1}{a} + \frac{1}{4} \right) \right\} \int_{X_{\alpha a}} \|\phi\|^2 dv_g, \end{aligned}$$

which yields

$$\int_{Y_{2a} \cap X_{\alpha a}} \|\phi\|^2 dv_g \leq 2C^{-1}e^{-2a} \left\{ \left(\frac{1}{a} + \frac{1}{a^2} \right) + \mu_n(\Delta_*^p|_{X_{\alpha a}}) \left(\frac{1}{a} + \frac{3}{4} \right) \right\} \int_{X_{\alpha a}} \|\phi\|^2 dv_g.$$

By **Lemma 3.5**,

$$\mu_n(\Delta_*^p|_{X_{\alpha a}}) \leq \mu_n(\Delta_{dir}^p|_{X_{a_0}})$$

and we have proved the proposition. □

A proof of Theorem 3.1

As before let φ_i be an eigenvector of Δ^p whose eigenvalue is $\mu_i(\Delta^p)$ and $\mathfrak{M}_{n,\chi}$ an n -dimensional subspace of $D(\Delta_{dir}^p|_{X_{2a}})$ spanned by $\{\chi\varphi_1, \dots, \chi\varphi_n\}$. Let us choose $\varphi \in \mathfrak{M}_n$ to be

$$\frac{\int_X \|\nabla(\chi\varphi)\|^2 dv_g}{\int_X \|\chi\varphi\|^2 dv_g} = \sup_{f \in \mathfrak{M}_{n,\chi}} \frac{\int_X \|\nabla f\|^2 dv_g}{\int_X \|f\|^2 dv_g}.$$

By **Lemma 3.3** the RHS is greater than or equal to $\mu_n(\Delta_{dir}^p|_{X_{2a}})$ and therefore

$$\int_X \|\nabla(\chi\varphi)\|^2 dv_g \geq \mu_n(\Delta_{dir}^p|_{X_{2a}}) \int_X \|\chi\varphi\|^2 dv_g.$$

On the other hand by a choice of χ ,

$$\begin{aligned} \|\nabla(\chi\varphi)\|^2 &\leq \|\nabla\chi \cdot \varphi\|^2 + 2|\operatorname{Re}(\nabla\chi \cdot \varphi, \chi\nabla\varphi)| + \|\chi\nabla\varphi\|^2 \\ &\leq \frac{1}{a^2}\|\varphi\|^2 + \frac{2}{a}|\operatorname{Re}(\varphi, \nabla\varphi)| + \|\nabla\varphi\|^2 \\ &\leq \left(\frac{1}{a^2} + \frac{1}{a}\right)\|\varphi\|^2 + \left(\frac{1}{a} + 1\right)\|\nabla\varphi\|^2 \end{aligned}$$

hence

$$\begin{aligned} &\left(\frac{1}{a^2} + \frac{1}{a}\right) \int_X \|\varphi\|^2 dv_g + \left(\frac{1}{a} + 1\right) \int_X \|\nabla\varphi\|^2 dv_g \\ &\geq \mu_n(\Delta_{dir}^p|_{X_{2a}}) \int_X \|\chi\varphi\|^2 dv_g \\ &\geq \mu_n(\Delta_{dir}^p|_{X_{2a}}) \int_{X_a} \|\varphi\|^2 dv_g \\ &= \mu_n(\Delta_{dir}^p|_{X_{2a}}) \left(\int_X \|\varphi\|^2 dv_g - \int_{Y_a} \|\varphi\|^2 dv_g \right). \end{aligned}$$

Here by (24) the most left hand side is less than or equal to

$$\left\{ \left(\frac{1}{a} + 1\right)\mu_n(\Delta^p) + \left(\frac{1}{a^2} + \frac{1}{a}\right) \right\} \int_X \|\varphi\|^2 dv_g,$$

and by **Proposition 3.5** the most right hand side is greater than or equal to

$$\mu_n(\Delta_{dir}^p|_{X_{2a}}) \left(1 - \rho_n\left(\frac{a}{2}\right)\right) \int_X \|\varphi\|^2 dv_g.$$

Thus we have obtained

$$\left(\frac{1}{a} + 1\right)\mu_n(\Delta^p) \geq \mu_n(\Delta_{dir}^p|_{X_{2a}}) \left(1 - \rho_n\left(\frac{a}{2}\right)\right) - \left(\frac{1}{a^2} + \frac{1}{a}\right).$$

Now since

$$\lim_{a \rightarrow \infty} \rho_n\left(\frac{a}{2}\right) = 0,$$

and **Lemma 3.5** implies the desired result.

□

A proof of Theorem 3.2.

Since a proof is almost same as one of **Theorem 3.1** we will only indicate where a modification is necessary. As before let ϕ_i be an eigenform of $\Delta_*^p|_{X_{3a}}$ whose eigenvalue is $\mu_i(\Delta_*^p|_{X_{3a}})$ and $\mathfrak{M}_n(3a)_\chi$ a n -dimensional subspace of $D(\Delta_{dir}^p|_{X_{2a}})$ spanned by $\{\chi\phi_1, \dots, \chi\phi_n\}$. We choose $\phi \in \mathfrak{M}_n(3a)$ so that

$$\frac{\int_{X_{3a}} \|\nabla(\chi\phi)\|^2 dv_g}{\int_{X_{3a}} \|\chi\phi\|^2 dv_g} = \sup_{f \in \mathfrak{M}_n(3a)_\chi} \frac{\int_X \|\nabla f\|^2 dv_g}{\int_X \|f\|^2 dv_g}.$$

Then **Lemma 3.3** implies

$$\int_{X_{3a}} \|\nabla(\chi\phi)\|^2 dv_g \geq \mu_n(\Delta_{dir}^p|_{X_{2a}}) \int_{X_{3a}} \|\chi\phi\|^2 dv_g.$$

Using (25) and **Proposition 3.6** instead (24) and **Proposition 3.5**, respectively the same computation as in **Theorem 3.1** will yield

$$\left(\frac{1}{a} + 1\right) \mu_n(\Delta_*^p|_{X_{3a}}) \geq \mu_n(\Delta_{dir}^p|_{X_{2a}}) \left(1 - \rho_n^0\left(\frac{a}{2}\right)\right) - \left(\frac{1}{a^2} + \frac{1}{a}\right).$$

By **Lemma 3.5** $\mu_n(\Delta_*^p|_{X_{3a}})$ is bounded by $\mu_n(\Delta_{dir}^p|_{X_{3a}})$ from above. Therefore

$$\lim_{a \rightarrow \infty} \mu_n(\Delta_*^p|_{X_{3a}}) = \lim_{a \rightarrow \infty} \mu_n(\Delta_{dir}^p|_{X_{3a}}),$$

and this is equal to $\mu_n(\Delta^p)$ by **Theorem 3.1**.

□

3.4 A theorem of Cheeger-Müller type

Since both Δ^p and $\Delta_{abs}^p|_{X_a}$ have only discrete spectrum the p -th cohomology groups $H^p(X_a, \rho)$ and $H^p(X, \rho)$ is isomorphic to $\text{Ker } \Delta_{abs}^p|_{X_a}$ and $\text{Ker } \Delta_X^p$ by Hodge theory, respectively. If a is sufficiently large, $H^p(X_a, \rho)$ is isomorphic to $H^p(X, \rho)$ by the restriction and therefore $\text{Ker } \Delta_X^p$ is also isomorphic to $\text{Ker } \Delta_{abs}^p|_{X_a}$. Since ρ is cuspidal $H^0(T_\nu, \rho) = 0$ for every ν and

$$H^2(T_\nu, \rho) = 0,$$

by Poincaré duality. Because ρ is a local system on a flat torus T_ν we see

$$H^1(T_\nu, \rho) = 0,$$

by the index theorem. Therefore the exact sequence

$$\begin{aligned} H^{p-1}(\partial X_a, \rho) &= \oplus_{\nu=1}^h H^{p-1}(T_\nu, \rho) \rightarrow H^p(X_a, \partial X_a, \rho) \rightarrow H^p(X_a, \rho) \\ \rightarrow H^p(\partial X_a, \rho) &= \oplus_{\nu=1}^h H^p(T_\nu, \rho) \end{aligned}$$

shows $H^p(X_a, \partial X_a, \rho)$ is isomorphic to $H^p(X_a, \rho)$. Thus we find

$$h^p(X, \rho) = h^{3-p}(X, \rho),$$

by Poincaré duality. (Recall $h^p(X, \rho)$ is the dimension of $H^p(X_a, \rho)$.) In particular since $h^0(X, \rho)$ vanishes so does $h^3(X, \rho)$. Moreover Hodge $*$ operator yields an isomorphism

$$\text{Ker } \Delta_{abs}^p|_{X_a} \stackrel{*}{\simeq} \text{Ker } \Delta_{rel}^{3-p}|_{X_a},$$

and therefore

$$h^p(X, \rho) = \dim \text{Ker } \Delta_{abs}^p|_{X_a} = \text{Ker } \Delta_{rel}^p|_{X_a}.$$

A *partial spectral zeta function* of $\Delta_*^p|_{X_a}$ ($*$ = *abs* or *rel*) is defined to be

$$\zeta_{X_a,*}^{(p)}(z, \rho) = \frac{1}{\Gamma(z)} \int_0^\infty \theta_p(t, a) t^{z-1} dt, \quad \theta_p(t, a) = \text{Tr}[e^{-t\Delta_*^p|_{X_a}}] - h^p(X, \rho).$$

(Here a is assumed to be sufficiently large.) If $\text{Re } z$ is sufficiently large it absolutely converges. Since $\theta_p(t, a)$ has an asymptotic expansion

$$\theta_p(t, a) \sim t^{-\frac{3}{2}} \sum_{l=0}^{2N} c_l(a) t^{l/2} + O(t^{N-\frac{3}{2}}), \quad \text{as } t \rightarrow 0. \quad (26)$$

the same argument as the beginning of this section will show that $\zeta_{X_a,*}^{(p)}(z, \rho)$ is meromorphically continued to the whole plane and that it is regular at the origin. We define *the spectral zeta function* of X_a as

$$\zeta_{X_a}(z, \rho) = \sum_{p=0}^3 (-1)^p p \cdot \zeta_{X_a,abs}^{(p)}(z, \rho).$$

Since Hodge $*$ operator commutes with Hodge Laplacian and since it interchanges two boundary conditions,

$$\zeta_X^{(p)}(z, \rho) = \zeta_X^{(3-p)}(z, \rho), \quad \zeta_{X_a,abs}^{(p)}(z, \rho) = \zeta_{X_a,rel}^{(3-p)}(z, \rho).$$

Therefore

$$\zeta_{X_a}(z, \rho) = 2\zeta_{X_a,rel}^{(1)}(z, \rho) - \zeta_{X_a,abs}^{(1)}(z, \rho) - 3\zeta_{X_a,rel}^{(0)}(z, \rho)$$

and

$$\zeta_X(z, \rho) = \zeta_X^{(1)}(z, \rho) - 3\zeta_X^{(0)}(z, \rho).$$

Theorem 3.3.

$$\lim_{a \rightarrow \infty} \zeta_{X_a}(z, \rho) = \zeta_X(z, \rho).$$

Corollary 3.2.

$$\lim_{a \rightarrow \infty} \zeta'_{X_a}(0, \rho) = \zeta'_X(0, \rho).$$

Proof of Theorem 3.3. Let us write

$$\begin{aligned}\int_0^\infty \theta_p(t) t^{z-1} dt &= \int_0^1 \theta_p(t) t^{z-1} dt + \int_1^\infty \theta_p(t) t^{z-1} dt, \\ \int_0^\infty \theta_p(t, a) t^{z-1} dt &= \int_0^1 \theta_p(t, a) t^{z-1} dt + \int_1^\infty \theta_p(t, a) t^{z-1} dt\end{aligned}$$

We will investigate convergence of corresponding terms in RHS. For the second term **Theorem 3.2** implies

$$\lim_{a \rightarrow \infty} \int_1^\infty \theta_p(t, a) t^{z-1} dt = \int_1^\infty \theta_p(t) t^{z-1} dt.$$

As we have seen at the begining of this section $\theta_p(t)$ has an asymptotic expansion (see (14)),

$$\theta_p(t) \sim t^{-\frac{3}{2}} \sum_{l=0}^{2N} c_l t^{l/2} + O(t^{N-\frac{3}{2}}), \quad c_{2l+1} = 0,$$

around $t = 0$. We put

$$\theta_p(t, a)^{(N)} = \theta_p(t, a) - t^{-\frac{3}{2}} \sum_{l=0}^{2N} c_l(a) t^{l/2}, \quad \theta_p(t)^{(N)} = \theta_p(t) - t^{-\frac{3}{2}} \sum_{l=0}^{2N} c_l t^{l/2}.$$

Then $t^{\frac{3}{2}-N} \theta_p(t, a)^{(N)}$ and $t^{\frac{3}{2}-N} \theta_p(t)^{(N)}$ are bounded on $(0, 1]$. By **Theorem 3.2** we see

$$\lim_{a \rightarrow \infty} c_l(a) = c_l, \quad 0 \leq l \leq 2N,$$

which implies in turn

$$\lim_{a \rightarrow \infty} \theta_p(t, a)^{(N)} = \theta_p(t)^{(N)} \quad \text{on } (0, 1]$$

Therefore if $\operatorname{Re} z > \frac{3}{2} - N$,

$$\lim_{a \rightarrow \infty} \int_0^1 \theta_p(t, a)^{(N)} t^{z-1} dt = \int_0^1 \theta_p(t)^{(N)} t^{z-1} dt.$$

and

$$\int_0^1 \theta_p(t, a) t^{z-1} dt = \sum_{l=0}^{2N} \frac{c_l(a)}{z + \frac{l-3}{2}} + \int_0^1 \theta_p(t, a)^{(N)} t^{z-1} dt,$$

converges to

$$\int_0^1 \theta_p(t) t^{z-1} dt = \sum_{l=0}^{2N} \frac{c_l}{z + \frac{l-3}{2}} + \int_0^1 \theta_p(t)^{(N)} t^{z-1} dt,$$

as $a \rightarrow \infty$.

□

For a finite dimensional vector space V we set

$$\det V = \wedge^{\dim V} V.$$

The *determinant* of a bounded complex of finite dimensional vector spaces (C^\cdot, ∂) is defined to be

$$\det(C^\cdot, \partial) = \otimes_i (\det C^i)^{(-1)^i}.$$

Here for a one dimensional complex vector space L , L^{-1} is its dual. Due to Knudsen and Mumford, there is a canonical isomorphism

$$\det(C^\cdot, \partial) \simeq \otimes_i \det H^i(C^\cdot, \partial)^{(-1)^i}. \quad (27)$$

Let $\Sigma = \{\Sigma_p\}_p$ be a triangulation of X_a where Σ_p is the set of p -simplices and $\mathbf{e} = \{\mathbf{e}_1, \dots, \mathbf{e}_r\}$ a unitary basis of ρ . We define a Hermitian inner product on the group of p -cochains:

$$C^p(\Sigma, \rho) = C^p(\Sigma) \otimes \rho,$$

so that $\{[\sigma]^* \otimes \mathbf{e}_i\}$ form its unitary basis, where $[\sigma]^*$ is the dual vector of $[\sigma]$. Now the Knudsen and Mumford isomorphism induces a metric $\|\cdot\|_{FR,a}$ on $\det H^\cdot(X_a, \rho) = \otimes_i \det H^i(C^\cdot(\Sigma, \rho))^{(-1)^i}$, which is referred as *Franz-Reidemeister metric*. Via the isomorphism

$$H^\cdot(X_a, \rho) \simeq H^\cdot(X, \rho),$$

it yields a metric $\|\cdot\|_{FR,a}$ on $\det H^\cdot(X, \rho)$. Notice that they are independent of a as far as it is sufficiently large since we can use the same triangulations to define them. Therefore the limit

$$\|\cdot\|_{FR} = \lim_{a \rightarrow \infty} \|\cdot\|_{FR,a}$$

is well-defined. For a later purpose we will describe it in terms of a combinatoric zeta function.

A triangulation Σ of X_a induces a simplicial decomposition $\tilde{\Sigma}$ of the universal covering \tilde{X}_a . Let $\{\sigma_1^{(p)}, \dots, \sigma_{\gamma_p}^{(p)}\}$ the set of p -simplices of $\tilde{\Sigma}$ which are a lift of Σ_p . Then $C_p(\tilde{\Sigma})$ is a free $\mathbb{C}[\Gamma]$ -module generated by these elements. A twisted chain complex is defined to be

$$C_\cdot(\Sigma, \rho) = C_\cdot(\tilde{\Sigma}) \otimes_{\mathbb{C}[\Gamma]} \rho,$$

which is a bounded complex of finite dimensional vector spaces. We will introduce a Hermitian inner product so that $\{\sigma_i^{(p)} \otimes \mathbf{e}_j\}$ is a unitary basis. Here is

an explicit description of the boundary map: Let $\sum_k (-1)^k \gamma_k [\sigma_k^{(p-1)}] (\gamma_k \in \Gamma)$ be the boundary of $[\sigma_i^{(p)}] \in C^p(\tilde{\Sigma})$. Then

$$\partial([\sigma_i^{(p)}] \otimes \mathbf{e}_j) = \sum_k (-1)^k [\sigma_k^{(p-1)}] \otimes \rho(\gamma_k) \mathbf{e}_j.$$

Let $(C(\Sigma, \rho), \delta)$ be the dual complex. By the inner product we may identify $C(\Sigma, \rho)$ with $C_p(\Sigma, \rho)$ and in particular the dual vector of $[\sigma_i^{(p)}] \otimes \mathbf{e}_j$ will be identified with itself. Thus $(C(\Sigma, \rho), \delta)$ is a complex such that each $C^p(\Sigma, \rho)$ is nothing but $C_p(\Sigma, \rho)$ as a vector space but the differential δ is the Hermitian dual of ∂ . Let us define a (positive) combinatoric Laplacian Δ_{comb}^p on $C^p(\Sigma, \rho) (= C_p(\Sigma, \rho))$ to be

$$\Delta_{comb}^p = \partial\delta + \delta\partial.$$

Then

$$H^p(X_a, \rho) (= H_p(X_a, \rho)) = \text{Ker}[\Delta_{comb}^p],$$

has the inner product $(\cdot, \cdot)_{l^2, X_a}$ induced by the metric on $C_p(\Sigma, \rho)$ (Here we have identified $H^p(X_a, \rho)$ with $\text{Ker}[\Delta_{comb}^p]$ which is a subspace of $C_p(\Sigma, \rho)$). It induces a norm $|\cdot|_{l^2, X_a}$ on the determinant $\otimes_p \det H^p(X_a, \rho)^{(-1)^p}$. Let us define the combinatoric zeta function to be

$$\zeta_{comb}(s, X_a) = \sum_p (-1)^p p \cdot \zeta_{comb}^{(p)}(s, X_a),$$

where

$$\zeta_{comb}^{(p)}(s, X_a) = \sum_{\lambda} \lambda^{-s}.$$

Here λ runs through positive eigenvalues of Δ_{comb}^p on $C^p(\Sigma, \rho)$. The modified Franz-Reidemeister torsion $\tau^*(X_a, \rho)$ is defined as

$$\tau^*(X_a, \rho) = \exp(-\frac{1}{2} \zeta'_{comb}(0, \rho)).$$

If $H^1(X, \rho)$ vanishes so does every $H^p(X, \rho)$ and $\tau^*(X_a, \rho)$ is the usual Franz-Reidemeister torsion $\tau(X_a, \rho)$ ([15]). It is known that $\|\cdot\|_{FR, a}$ is equal to ([2][15])

$$|\cdot|_{l^2, X_a} \cdot \tau^*(X_a, \rho).$$

By construction since both $|\cdot|_{l^2, X_a}$ and $\tau^*(X_a, \rho)$ depend only on a triangulation Σ , they are independent of sufficiently large a as before. Thus the limit

$$|\cdot|_{l^2, X} = \lim_{a \rightarrow \infty} |\cdot|_{l^2, X_a}, \quad \tau^*(X, \rho) = \lim_{a \rightarrow \infty} \tau^*(X_a, \rho),$$

is well-defined and we set

$$\|\cdot\|_{FR} = |\cdot|_{l^2, X} \cdot \tau^*(X, \rho).$$

On the other hand since $H^p(X_a, \rho)$ is isomorphic to

$$\text{Ker} \Delta_{abs}^p|_{X_a} \subset L^2(X_a, \Omega^p(\rho))$$

the inner product on $L^2(X_a, \Omega^p(\rho))$ induces a metric on $H^p(X_a, \rho)$. Thus by the isomorphism $H^p(X, \rho) \simeq H^p(X_a, \rho)$ we have a norm $|\cdot|_{L^2, X_a}$ on $\det H^p(X, \rho)$. The Ray-Singer metric $\|\cdot\|_{RS, a}$ is defined to be

$$\|\cdot\|_{RS, a} = |\cdot|_{L^2, X_a} \cdot \exp\left(-\frac{1}{2}\zeta'_{X_a}(0, \rho)\right).$$

Similary using the canonical isomorphism

$$H^p(X, \rho) \simeq \text{Ker} \Delta^p \subset L^2(X, \Omega^p(\rho)),$$

Ray-Singer metric $\|\cdot\|_{RS}$ on $\det H^p(X, \rho)$ is defined as

$$\|\cdot\|_{RS} = |\cdot|_{L^2, X} \cdot \exp\left(-\frac{1}{2}\zeta'_X(0, \rho)\right).$$

Then we will show

Theorem 3.4. $\|\cdot\|_{FR}$ and $\|\cdot\|_{RS}$ coincide. In particular

$$\exp(-\zeta'_X(0, \rho)) = \left(\frac{|\cdot|_{L^2, X}}{|\cdot|_{L^2, X_a}}\right)^2 \tau^*(X, \rho)^2.$$

Proposition 3.7.

$$\lim_{a \rightarrow \infty} |\cdot|_{L^2, X_a} = |\cdot|_{L^2, X}.$$

In fact for a sufficiently large a let $\{\xi_{a,i}\}_i$ be an orthonormal base of $\text{Ker} \Delta_{abs}^p|_{X_a}$ and we define a map

$$\text{Ker} \Delta^p \xrightarrow{P_a} \text{Ker} \Delta_{abs}^p|_{X_a}$$

as

$$P_a \psi = \sum_i \int_{X_a} (\psi, \xi_{a,i}) dv_g \cdot \xi_{a,i}.$$

Then we claim the following.

Lemma 3.6.

$$\lim_{a \rightarrow \infty} \int_{X_a} \|\psi - P_a \psi\|^2 dv_g = 0.$$

The following corollary will imply **Proposition 3.7**.

Corollary 3.3. For $\psi \in \text{Ker} \Delta^p$,

$$\lim_{a \rightarrow \infty} \int_{X_a} \|P_a \psi\|^2 dv_g = \int_X \|\psi\|^2 dv_g.$$

Proof. Immediately from **Proposition 3.5** and **Lemma 3.6**.

□

Proof of Lemma 3.6. In the following arguments all C are positive constants independent of a . Let ϕ_λ be an eigenform of $\Delta_{abs}^p|_{X_a}$ whose eigenvalue is λ and that

$$\int_{X_a} \|\phi_\lambda\|^2 dv_g = 1.$$

Let us expand ψ as

$$\psi = \sum_{\lambda} \int_{X_a} (\psi, \phi_\lambda) dv_g \cdot \phi_\lambda.$$

Since

$$\int_{X_a} \|\psi - P_a \psi\|^2 dv_g = \sum_{\lambda > 0} \left| \int_{X_a} (\psi, \phi_\lambda) dv_g \right|^2,$$

it is sufficient to show that for $\phi = \phi_\lambda$ ($\lambda > 0$),

$$\left| \int_{X_a} (\psi, \phi) dv_g \right| \leq C e^{-a} \left(\int_{X_a} \|\psi\|^2 dv_g + C \right).$$

Let us choose $\chi \in C_c^\infty(X_a)$ so that

1. $0 \leq \chi \leq 1$.
2. $|\nabla \chi|, |\Delta \chi|$ are bounded by 1.
3. $\chi|_{X_{a/2}} = 1$.

By Stokes theorem,

$$\int_{X_a} (\Delta^p(\chi\psi), \phi) dv_g = \int_{X_a} (\chi\psi, \Delta^p\phi) dv_g = \lambda \int_{X_a} \chi(\psi, \phi) dv_g \quad (28)$$

Since $\Delta^p\psi = 0$ and by the property 3 of χ , LHS of (28) becomes

$$\begin{aligned} \int_{X_a} (\Delta^p(\chi\psi), \phi) dv_g &= \int_{X_a} (\Delta\chi \cdot \psi, \phi) dv_g + 2 \int_{X_a} (\nabla\chi \cdot \nabla\psi, \phi) dv_g \\ &= \int_{Y_{a/2} \cap X_a} (\Delta\chi \cdot \psi, \phi) dv_g + 2 \int_{Y_{a/2} \cap X_a} (\nabla\chi \cdot \nabla\psi, \phi) dv_g. \end{aligned}$$

Let us consider the first term. Using the property 2 of χ the Schwartz inequality implies

$$\begin{aligned} \left| \int_{Y_{a/2} \cap X_a} (\Delta\chi \cdot \psi, \phi) dv_g \right| &\leq \frac{1}{2} \left(\int_{Y_{a/2} \cap X_a} \|\psi\|^2 dv_g + \int_{Y_{a/2} \cap X_a} \|\phi\|^2 dv_g \right) \\ &\leq \frac{1}{2} \left(\int_{Y_{a/2}} \|\psi\|^2 dv_g + \int_{Y_{a/2} \cap X_a} \|\phi\|^2 dv_g \right). \end{aligned}$$

By **Proposition 3.5**,

$$\begin{aligned}
\int_{Y_{a/2}} \|\psi\|^2 dv_g &\leq C e^{-a/2} \int_X \|\psi\|^2 dv_g \\
&= C e^{-a/2} \left(\int_{X_a} \|\psi\|^2 dv_g + \int_{Y_a} \|\psi\|^2 dv_g \right) \\
&\leq C e^{-a/2} \int_{X_a} \|\psi\|^2 dv_g + C e^{-a/2} \int_{Y_{a/2}} \|\psi\|^2 dv_g,
\end{aligned}$$

and therefore changing C we obtain

$$\int_{Y_{a/2}} \|\psi\|^2 dv_g \leq C e^{-a/2} \int_{X_a} \|\psi\|^2 dv_g.$$

Using **Proposition 3.6** instead **Proposition 3.5** the same computation will show

$$\int_{Y_{a/2} \cap X_a} \|\phi\|^2 dv_g \leq C e^{-a/2} \int_{X_a} \|\phi\|^2 dv_g = C e^{-a/2} \quad (29)$$

and thus

$$\left| \int_{Y_{a/2} \cap X_a} (\Delta \chi \cdot \psi, \phi) dv_g \right| \leq C e^{-a/2} \left(\int_{X_a} \|\psi\|^2 dv_g + C \right).$$

Next we will estimate the second term. Using the property 2 of χ , the Schwartz inequality yields,

$$|2 \int_{X_a} (\nabla \chi \cdot \nabla \psi, \phi) dv_g| \leq \int_{Y_{a/2}} \|\nabla \psi\|^2 dv_g + \int_{Y_{a/2} \cap X_a} \|\phi\|^2 dv_g.$$

Since

$$\int_{Y_{a/2}} \|\nabla \psi\|^2 dv_g \leq \int_X \|\nabla \psi\|^2 dv_g = \int_X (\psi, \Delta^p \psi) dv_g = 0,$$

and by (29), we obtain

$$|2 \int_{X_a} (\nabla \chi \cdot \nabla \psi, \phi) dv_g| \leq C e^{-a/2}.$$

Thus LHS of (28) is bounded by $C e^{-a/2} (\int_{X_a} \|\psi\|^2 dv_g + C)$. Let us consider RHS of (28). The property 3 implies

$$\int_{X_a} \chi(\psi, \phi) dv_g = \int_{X_{a/2}} (\psi, \phi) dv_g + \int_{Y_{a/2} \cap X_a} \chi(\psi, \phi) dv_g.$$

But by the previous arguments

$$\begin{aligned}
\left| \int_{Y_{a/2} \cap X_a} \chi(\psi, \phi) dv_g \right| &\leq \int_{Y_{a/2} \cap X_a} |(\psi, \phi)| dv_g \\
&\leq \frac{1}{2} \left\{ \int_{Y_{a/2}} \|\psi\|^2 dv_g + \int_{Y_{a/2} \cap X_a} \|\phi\|^2 dv_g \right\} \\
&\leq C e^{-a/2} \left(\int_{X_a} \|\psi\|^2 dv_g + C \right),
\end{aligned}$$

and we will obtain

$$|\int_{X_{a/2}} (\psi, \phi) dv_g| \leq C e^{-a/2} (\int_{X_a} \|\psi\|^2 dv_g + C).$$

Notice that

$$\begin{aligned} |\int_{X_a} (\psi, \phi) dv_g - \int_{X_{a/2}} (\psi, \phi) dv_g| &= |\int_{X_a \cap Y_{a/2}} (\psi, \phi) dv_g| \\ &\leq \int_{X_a \cap Y_{a/2}} |(\psi, \phi)| dv_g \\ &\leq C e^{-a/2} (\int_{X_a} \|\psi\|^2 dv_g + C), \end{aligned}$$

and the desired result has been obtained since

$$\begin{aligned} |\int_{X_a} (\psi, \phi) dv_g| &\leq |\int_{X_a} (\psi, \phi) dv_g - \int_{X_{a/2}} (\psi, \phi) dv_g| + |\int_{X_{a/2}} (\psi, \phi) dv_g| \\ &\leq C e^{-a/2} (\int_{X_a} \|\psi\|^2 dv_g + C). \end{aligned}$$

□

Let us choose a sufficiently large number a and small positive number δ . Let g_0 be a Riemannian metric on X such that

$$g_0(x) = \begin{cases} g(x) & \text{if } x \in X_{a-\delta} \\ du^2 + e^{-2a}(dx^2 + dy^2) & \text{if } x \in Y_a \end{cases}$$

We will consider a one parameter family of metrics:

$$g_q = (1 - q)g_0 + qg, \quad 0 \leq q \leq 1.$$

Let $\{\mathbf{e}^0, \mathbf{e}^1, \mathbf{e}^2\}$ be an orthonormal frame of $\Omega^1|_{\partial X_a}$ with respect to $g(q)|_{\partial X_a} = du^2 + e^{-2a}(dx^2 + dy^2)$ so that $\mathbf{e}^0 = du$. Let $h(q)$ and $R(q)$ be the second fundamental of ∂X_a with respect to $g(q)$ and the curvature tensor of $g(q)$, respectively. Then we define elements

$$\hat{h}(q) = \sum_{1 \leq a, b \leq 2} h(q)_{ab} \mathbf{e}^a \otimes \mathbf{e}^b$$

and

$$\hat{R}_0(q) = \frac{1}{4} \sum_{j, k, l} R(q)_{0jkl} \mathbf{e}^j \otimes (\mathbf{e}^k \wedge \mathbf{e}^l)$$

of $\Omega|_{\partial X_a} \otimes \Omega|_{\partial X_a}$. Using Berezin integral [2], \int^B , we put

$$\phi_a = \int_0^1 dq \int^B \hat{h}(q) \hat{R}_0(q) \in \Omega|_{\partial X_a}.$$

Fact 3.3. ([3])

$$\log \left(\frac{\|\cdot\|_{RS,a}}{\|\cdot\|_{FR,a}} \right) = \chi(\partial X_a, \rho) \log 2 + \gamma \cdot r \int_{\partial X_a} \phi_a,$$

where γ is an absolute constant.

Notice that the term $\tilde{e}(g_0, g_q)$ in the original formula vanishes because the dimension of X is three. A straightforward computation will show that the norm of ϕ_a is bounded by a constant C which is independent of a . Thus,

$$\left| \int_{\partial X_a} \phi_a \right| \leq C \cdot \text{vol}(\partial X_a) \leq C' \cdot e^{-2a},$$

where C' is also independent of a . Since ρ is a unitary local system on ∂X_a which is a disjoint union of flat tori, $\chi(\partial X_a, \rho)$ vanishes by the index theorem. Therefore we have shown

Proposition 3.8.

$$\lim_{a \rightarrow \infty} \|\cdot\|_{RS,a} = \|\cdot\|_{FR}.$$

Proof of Theorem 3.4. By **Proposition 3.8**

$$\|\cdot\|_{FR} = \lim_{a \rightarrow \infty} \{|\cdot|_{L^2, X_a} \cdot \exp(-\frac{1}{2} \zeta'_{X_a}(0, \rho))\}.$$

But by **Proposition 3.7** and **Corollary 3.2** this is equal to $\|\cdot\|_{RS}$.

□

3.5 A computation of the leading coefficient

We will interpret the ratio $|\cdot|_{L^2, X} / |\cdot|_{L^2, X}$ as a period. Hereafter we will identify $H^p(X, \rho)$ and $\text{Ker} \Delta_X^p$ by Hodge theory. Let $\phi^{(p)} = \{\phi_1^{(p)}, \dots, \phi_{h^p(X, \rho)}^{(p)}\}$ and $\psi^{(p)} = \{\psi_1^{(p)}, \dots, \psi_{h^p(X, \rho)}^{(p)}\}$ be its unitary basis with respect to $(\cdot, \cdot)_{L^2, X}$ and $(\cdot, \cdot)_{L^2, X}$, respectively. Then $\phi^{(p)}$ determines the dual basis $\phi_{(p)} = \{\phi_{(p), i}\}_{1 \leq i \leq h^p(X, \rho)}$ of $H_p(X, \rho)$ and we write

$$\psi_i^{(p)} = \sum_{j=1}^{h^p(X, \rho)} \int_{\phi_{(p), j}} \psi_i^{(p)} \cdot \phi_j^{(p)}.$$

A period matrix of twisted p -forms is defined to be

$$P(X, \rho)_p = \left(\int_{\phi_{(p), j}} \psi_i^{(p)} \right)_{ij}.$$

We call an alternating product $\prod_p |\det P(X, \rho)_p|^{(-1)^p}$ a period of (X, ρ) and will denote it by $\text{Per}(X, \rho)$. A simple computation shows

$$\psi_1^{(p)} \wedge \dots \wedge \psi_{h^p(X, \rho)}^{(p)} = \det P(X, \rho)_p \cdot \phi_1^{(p)} \wedge \dots \wedge \phi_{h^p(X, \rho)}^{(p)}.$$

and by definition we have

$$|\otimes_p (\psi_1^{(p)} \wedge \cdots \wedge \psi_{h^p(X,\rho)}^{(p)})^{(-1)^p}|_{L^2, X} = |\otimes_p (\phi_1^{(p)} \wedge \cdots \wedge \phi_{h^p(X,\rho)}^{(p)})^{(-1)^p}|_{l^2, X} = 1$$

Therefore

$$\begin{aligned} \frac{|\otimes_p (\psi_1^{(p)} \wedge \cdots \wedge \psi_{h^p(X,\rho)}^{(p)})^{(-1)^p}|_{l^2, X}}{|\otimes_p (\psi_1^{(p)} \wedge \cdots \wedge \psi_{h^p(X,\rho)}^{(p)})^{(-1)^p}|_{L^2, X}} &= |\otimes_p (\psi_1^{(p)} \wedge \cdots \wedge \psi_{h^p(X,\rho)}^{(p)})^{(-1)^p}|_{l^2, X} \\ &= \prod_p |\det P(X, \rho)_p|^{(-1)^p} \\ &= \text{Per}(X, \rho). \end{aligned}$$

Now using **Fact 3.1**, **Theorem 3.4** is reformulated as the following.

Theorem 3.5.

$$\lim_{z \rightarrow 0} z^{-2h^1(X,\rho)} R_X(z, \rho) = (\tau^*(X, \rho) \cdot \text{Per}(X))^2.$$

Corollary 3.4. *Suppose that $h^1(X, \rho)$ vanishes. Then*

$$R_X(0, \rho) = \tau(X, \rho)^2,$$

where $\tau(X, \rho)$ is the usual Franz-Reidemeister torsion.

3.6 An example

Let K be a knot in S^3 whose complement X_K admits a complete hyperbolic structure of finite volume and ρ a cuspidal unitary local system of rank r on X_K . Here the representaion of $\pi_1(X_K)$ associated to ρ is denoted by the same character. X_K is obtained by attaching 3-cells to a two dimensional CW-complex L which is a deformation retract of X_K . The argument of [11] **Lemma 7.2** will show the following.

Lemma 3.7.

$$\tau(X_K, \rho) = \tau(L, \rho).$$

We will compute $\tau(L, \rho)$. Let

$$\pi_1(X_K) = \langle x_1, \dots, x_n \mid r_1, \dots, r_{n-1} \rangle$$

be the Wirtinger presentation. Here $\{x_i\}_i$ (resp. $\{r_j\}_j$) is generators (resp. relators). $H_1(X_K, \mathbb{Z})$ is an infinite cyclic group and we fix a generator t . We choose x_i so that Hurewicz map

$$\pi_1(X_K) \xrightarrow{\epsilon} H_1(X_K, \mathbb{Z})$$

sends x_i to t . Then a group ring $\mathbb{C}[H_1(X_K, \mathbb{Z})]$ is isomorphic to Laurent polynomial ring $\Lambda = \mathbb{C}[t, t^{-1}]$ and a ring homomorphism:

$$\mathbb{C}[\pi_1(X_K)] \rightarrow \Lambda.$$

induced by Hurewicz map will be also denoted by ϵ . Also the representation ρ yields a homomorphism

$$\mathbb{C}[\pi_1(X_K)] \xrightarrow{\rho} M_r(\mathbb{C})$$

and let

$$\mathbb{C}[\pi_1(X_K)] \xrightarrow{\epsilon \otimes \rho} M_r(\Lambda).$$

be the tensor product of them. Composing this with the homomorphism induced by the natural projection from the free group F_n of n -generators to $\pi_1(X_K)$ we obtain,

$$\mathbb{C}[F_n] \xrightarrow{\Phi} M_r(\Lambda).$$

The set of 0-cells of L consists of only one point P_0 and one of 1-cells is

$$\{x_1, \dots, x_n\}.$$

In order to obtain the relation it is necessary to attach 2-cells

$$\{y_1, \dots, y_{n-1}\},$$

where y_j realizes the relator r_j . Let \tilde{L} be the universal covering of L and L_∞ an infinite cyclic covering which corresponds to $\text{Ker } \epsilon$. According to $p = 0$ (resp. $p = 1$ or $p = 2$), the p -th chain group $C_p(\tilde{L}, \mathbb{C})$ is a free right $\mathbb{C}[\pi_1(X_K)]$ module generated by P_0 (resp. $\{x_1, \dots, x_n\}$ or $\{y_1, \dots, y_{n-1}\}$) and the chain complex $C.(L_\infty, \rho)$ is defined to be

$$C_p(L_\infty, \rho) = C_p(\tilde{L}, \mathbb{C}) \otimes_{\mathbb{C}[\text{Ker } \epsilon]} \rho.$$

Thus the chain complex

$$C_2(L_\infty, \rho) \xrightarrow{\partial_2} C_1(L_\infty, \rho) \xrightarrow{\partial_1} C_0(L_\infty, \rho),$$

becomes

$$(\Lambda^{\oplus r})^{n-1} \xrightarrow{\partial_2} (\Lambda^{\oplus r})^n \xrightarrow{\partial_1} \Lambda^{\oplus r}, \quad (30)$$

and the differentials are described by Fox's free differential calculus. In fact it is known ([7]):

$$\partial_1 = \begin{pmatrix} \Phi(x_1 - 1) \\ \vdots \\ \Phi(x_n - 1) \end{pmatrix} = \begin{pmatrix} \rho(x_1)t - I_r \\ \vdots \\ \rho(x_n)t - I_r \end{pmatrix},$$

and

$$\partial_2 = \begin{pmatrix} \Phi(\frac{\partial r_1}{\partial x_1}) & \dots & \Phi(\frac{\partial r_1}{\partial x_n}) \\ \vdots & \ddots & \vdots \\ \Phi(\frac{\partial r_{n-1}}{\partial x_1}) & \dots & \Phi(\frac{\partial r_{n-1}}{\partial x_n}) \end{pmatrix}.$$

Here an each entry is an element of $M_r(\Lambda)$. $C_p(L_\infty, \rho)$ is considered as a space of row vectors and differentials act from the right. It is known that the determinant

of a certain entry of ∂_1 is not zero([19]). Therefore rearranging indices we may assume that $\det(\rho(x_n)t - I_r)$ is not zero and will denote it by $\Delta_0(t)$. Let us put

$$\Delta_1(t) = \det \begin{pmatrix} \Phi(\frac{\partial r_1}{\partial x_1}) & \cdots & \Phi(\frac{\partial r_1}{\partial x_{n-1}}) \\ \vdots & \ddots & \vdots \\ \Phi(\frac{\partial r_{n-1}}{\partial x_1}) & \cdots & \Phi(\frac{\partial r_{n-1}}{\partial x_{n-1}}) \end{pmatrix}.$$

Then *the twisted Alexander function* is defined to be ([7][8][19]),

$$\Delta_{K,\rho}(t) = \frac{\Delta_1(t)}{\Delta_0(t)}.$$

In the following we will assume $\Delta_1(t)$ is not zero. This implies that after tensored with $\mathbb{C}(t)$ (30) becomes acyclic. Thus $H_*(L_\infty, \rho)$ are torsion Λ -modules and in particular they are finite dimensional vector spaces. Let τ_i be the representation matrix of t . Remember that $C(L, \rho)$ is quasi-isomorphic to $C(X_K, \rho)$ and that

$$C(L, \rho) = C(L_\infty, \rho) \otimes_\Lambda \mathbb{C}.$$

Here \mathbb{C} is regarded as a Λ -module by $\mathbb{C} \simeq \Lambda/(t-1)$. Thus the exact sequence of complexes

$$0 \rightarrow C(L_\infty, \rho) \xrightarrow{(t-1)} C(L_\infty, \rho) \rightarrow C(L, \rho) \rightarrow 0,$$

induces

$$\begin{array}{ccccccc} 0 & \rightarrow & H_2(L_\infty, \rho) & \xrightarrow{\tau_2 - id} & H_2(L_\infty, \rho) & \rightarrow & H_2(X_K, \rho) \\ & & \rightarrow & H_1(L_\infty, \rho) & \xrightarrow{\tau_1 - id} & H_1(L_\infty, \rho) & \rightarrow H_1(X_K, \rho) \\ & & \rightarrow & H_0(L_\infty, \rho) & \xrightarrow{\tau_0 - id} & H_0(L_\infty, \rho) & \rightarrow H_0(X_K, \rho) \rightarrow 0. \end{array} \quad (31)$$

Since $h^0(X_K, \rho) = 0$, $H_0(X_K, \rho)$ vanishes by the universal coefficient theorem. Thus $\tau_0 - id$ is an isomorphism. Using the fact $\Delta_i(t)$ is a multiplication of the characteristic polynomial of τ_i and a certain unit of Λ , the following lemma is an easy consequence of (31).

Lemma 3.8. *The following are equivalent:*

1. $\Delta_{K,\rho}(1) \neq 0$.
2. $h^1(X_K, \rho) = 0$.
3. $h^1(X_K, \rho) = h^2(X_K, \rho) = 0$.

In [18] we have proved that the fact $h^i(X_K, \rho) = 0$ for all i implies

$$\tau(X_K, \rho) = |\Delta_{K,\rho}(1)|. \quad (32)$$

Thus **Corollary 3.4** and (32) prove the following.

Theorem 3.6. *Suppose $\Delta_{K,\rho}(1) \neq 0$. Then*

$$R_{X_K}(0, \rho) = |\Delta_{K,\rho}(1)|^2.$$

Here is an example of ρ such that a special value of Ruelle L-function at the origin can be computed explicitly. Let ξ be a complex number of modulus one. We define a morphism

$$H_1(X_K, \mathbb{Z}) \xrightarrow{\rho} U(1)$$

to be

$$\rho(t) = \xi.$$

Composing it with Hurewicz map we obtain a unitary character

$$\pi_1(X_K) \xrightarrow{\rho} U(1),$$

which yields a unitary local system of rank one on X_K . Notice that t represents a meridian of the boundary of a tubular neighborhood of K . Thus if $\xi \neq 1$, ρ is cuspidal. Moreover it is known ([7]§3.3):

$$\Delta_0(t) = 1 - \xi t, \quad \Delta_1(t) = A_K(\xi t),$$

where $A_K(t)$ is the Alexander polynomial. Now let us choose ξ so that $\xi \neq 1$ and that $A_K(\xi) \neq 0$. By **Theorem 3.6** we have the following.

Corollary 3.5.

$$R_{X_K}(0, \rho) = \left| \frac{A_K(\xi)}{1 - \xi} \right|^2.$$

4 A geometric meaning of coefficients

4.1 The K-group and regulators

Fact 4.1. ([6]) *Let*

$$K_{2n+1}(\mathbb{C}) \xrightarrow{r_{n+1}} \mathbb{R}$$

be the Borel regulator map. Then there is a natural element γ_X in $K_{2n+1}(\overline{\mathbb{Q}}) \otimes \mathbb{Q}$ such that

$$\text{vol}(X) = r_{n+1}(\gamma_X).$$

Thus **Theorem 2.2** shows if n is odd the ratio of the first and the second coefficient of the Taylor expansion at the origin is interpreted as an evaluation of the Borel regulator against a certain element of the algebraic K-group whereas if n is even a correction from cusps is necessary. It seems natural to expect the first coefficient also has such an interpretation. In fact it is true at least for a hyperbolic threefold, which will be explained below. Following [4] we will also explain how to construct γ_X for a closed hyperbolic threefold.

Let M be a smooth manifold and $P \rightarrow M$ a principal \mathbb{C}^\times -bundle with a flat connection. By Chern-Weil theory the image of the first Chern class $c_1(P) \in H^2(M, \mathbb{Z})$ in $H^2(M, \mathbb{C})$ vanishes. Thus the exact sequence

$$H^1(M, \mathbb{C}/\mathbb{Z}) \xrightarrow{\beta} H^2(M, \mathbb{Z}) \rightarrow H^2(M, \mathbb{C})$$

shows there is an element $\hat{c}_1(P) \in H^1(M, \mathbb{C}/\mathbb{Z})$ which maps to $c_1(P)$ by β . Let $\mathrm{GL}_1(\mathbb{C})^\delta$ be the multiplicative group \mathbb{C}^\times with the discrete topology and $B\mathrm{GL}_1(\mathbb{C})^\delta$ the classfying space. If we apply the previous construction to the universal flat \mathbb{C}^\times -bundle, we obtain an element \hat{c}_1 of $H^1(B\mathrm{GL}_1(\mathbb{C})^\delta, \mathbb{C}/\mathbb{Z})$. Since

$$H^1(B\mathrm{GL}_1(\mathbb{C})^\delta, \mathbb{C}/\mathbb{Z}) \simeq H^1(\mathrm{GL}_1(\mathbb{C}), \mathbb{C}/\mathbb{Z}) \simeq \mathrm{Hom}(\mathbb{C}^\times, \mathbb{C}/\mathbb{Z}),$$

\hat{c}_1 may be regarded as a homomorphism from $K_1(\mathbb{C}) \simeq \mathbb{C}^\times$ to \mathbb{C}/\mathbb{Z} . By definition r_1 is $\mathrm{Im}\hat{c}_1$ and it is known

$$\hat{c}_1(g) = \frac{i}{2\pi} \log g, \quad g \in \mathbb{C}^\times,$$

and thus $r_1(g) = \log |g|/2\pi$. Let us choose a unitary basis $\mathbf{e} = \{\mathbf{e}_1, \dots, \mathbf{e}_r\}$ of ρ . Suppose that $H^1(X, \rho)$ vanishes. Then as we have seen at the beginning of §3.4, $C(X, \rho)$ is acyclic. Following [12] a certain $\tau(X, \rho, \mathbf{e}) \in K_1(\mathbb{C})$ is defined which will be referred as *the Milnor element*. The Franz-Reidemeister torsion is nothing but its image in $K_1(\mathbb{C})/\mathrm{U}(1) \simeq \mathbb{R}$, i.e. its modulus. Thus

$$\begin{aligned} \log R_X(0, \rho) &= 2 \log \tau(X, \rho) \\ &= 2 \log |\tau(X, \rho, \mathbf{e})| \\ &= 4\pi r_1(\tau(X, \rho, \mathbf{e})). \end{aligned}$$

and we have found that $\log R_X(0, \rho)$ is interpreted as a period of the Milnor element by the first Borel regulator.

Taking account of an exceptional isomorphism

$$\mathrm{Spin}(3, 1) \simeq \mathrm{SL}_2(\mathbb{C}),$$

let us apply the previous construction to the universal flat bundle $\mathrm{SL}_2(\mathbb{C})$ -bundle on $B\mathrm{SL}_2(\mathbb{C})^\delta$. Then we will obtain the Chern-Simon class

$$\begin{aligned} \hat{c}_2 &\in H^3(B\mathrm{SL}_2(\mathbb{C})^\delta, \mathbb{C}/\mathbb{Z}) \simeq H^3(\mathrm{SL}_2(\mathbb{C}), \mathbb{C}/\mathbb{Z}) \\ &\simeq \mathrm{Hom}(H_3(\mathrm{SL}_2(\mathbb{C}), \mathbb{Z}), \mathbb{C}/\mathbb{Z}) \end{aligned}$$

which is a lift of the second Chern class $c_2 \in H^4(B\mathrm{SL}_2(\mathbb{C})^\delta, \mathbb{Z})$. In particular its imaginary part $\mathrm{Im}\hat{c}_2$ yields a \mathbb{Q} -linear map:

$$H_3(\mathrm{SL}_2(\mathbb{C}), \mathbb{Q}) \xrightarrow{\mathrm{Im}\hat{c}_2} \mathbb{R},$$

and it is known ([4])

$$\mathrm{Im}\hat{c}_2(g_1|g_2|g_3) = \frac{1}{4\pi^2} \mathrm{vol}(T(\infty, g_1\infty, g_1g_2\infty, g_1g_2g_3\infty)), \quad g_i \in \mathrm{SL}_2(\mathbb{C}).$$

Here $T(z_1, z_2, z_3, z_4)$ is a tetrahedron in \mathbb{H}^3 whose vertices are $\{z_1, z_2, z_3, z_4\}$ and edges are geodesics. Remember that the volume of an ideal tetrahedron is computed by *the Bloch-Wigner function*:

$$D(z) = \arg(1-z)\arg(z) + \operatorname{Im} \operatorname{Li}_2(z),$$

where $\operatorname{Li}_2(z)$ is *the dilogarithm*:

$$\operatorname{Li}_2(z) = \sum_{n=1}^{\infty} \frac{z^n}{n^2}.$$

More precisely it is known ([4])

$$\begin{aligned} \operatorname{vol}(T(\infty, g_1\infty, g_1g_2\infty, g_1g_2g_3\infty)) &= \operatorname{vol}(T(\infty, 0, 1, z(g_1, g_2, g_3))) \\ &= D(z(g_1, g_2, g_3)), \end{aligned}$$

where $z(g_1, g_2, g_3)$ is the cross ratio of $\{\infty, g_1\infty, g_1g_2\infty, g_1g_2g_3\infty\}$. Suppose that X is closed. Then since $H_3(B\operatorname{PSL}_2(\mathbb{C})^\delta, \mathbb{Q})$ is isomorphic to $H_3(B\operatorname{SL}_2(\mathbb{C})^\delta, \mathbb{Q})$ the natural inclusion $\Gamma \hookrightarrow \operatorname{SO}(3, 1) \simeq \operatorname{PSL}_2(\mathbb{C})$ induces

$$H_3(X, \mathbb{Q}) \simeq H_3(B\Gamma, \mathbb{Q}) \rightarrow H_3(B\operatorname{SL}_2(\mathbb{C})^\delta, \mathbb{Q}) \simeq H_3(\operatorname{SL}_2(\mathbb{C}), \mathbb{Q}).$$

Here recall that $H_3(\operatorname{SL}_2(\mathbb{C}), \mathbb{Z})$ is a direct summand of the Quillen's K-group $K_3(\mathbb{C})$. In fact ([4], (9.6)),

$$K_3(\mathbb{C}) \simeq H_3(\operatorname{SL}_2(\mathbb{C}), \mathbb{Z}) \oplus K_3^M(\mathbb{C}),$$

where $K_3^M(\mathbb{C})$ is the Milnor K-group. Thus the fundamental class $[X]$ of X defines an element γ_X in $K_3(\mathbb{C}) \otimes \mathbb{Q}$. According to Weil's rigidity Γ is conjugate a subgroup of $\operatorname{PSL}_2(\overline{\mathbb{Q}})$ and γ_X is contained in $K_3(\overline{\mathbb{Q}}) \otimes \mathbb{Q}$. We define *the second Borel regulator*

$$K_3(\mathbb{C}) \otimes \mathbb{Q} \xrightarrow{r_2} \mathbb{R}$$

to be a composition of $4\pi^2 \operatorname{Im} \hat{c}_2$ and the natural projection:

$$K_3(\mathbb{C}) \otimes \mathbb{Q} \rightarrow H_3(\operatorname{SL}_2(\mathbb{C}), \mathbb{Q}).$$

Then **Corollary 2.3** yields

$$\frac{d}{dz} \log R_X(z, \rho)|_{z=0} = -\frac{3r}{\pi} \cdot r_2(\gamma_X).$$

If X is not closed, $H_3(X)$ vanishes. So we have to use the relative homology group to define γ_X . (See [6] for details.) Thus we have found that the leading and the second coefficient of Taylor expansion of $R_X(z, \rho)$ at the origin are expressed by the logarithm and the dilogarithm, respectively.

4.2 The L^2 -torsion

The constant term of the logarithmic derivative of Ruelle L-function at the origin is also related to L^2 -analytic torsion ([1][9]). Following [1], we remember the von Neumann trace. Let ω_p the action of Γ on $L^2(\mathbb{H}^d, \Omega^p)$. Then $L^2(\mathbb{H}^d, \Omega^p \otimes \mathbb{C}^r)$ is a Γ -module by $\omega_p \otimes \rho$ and there is an isomorphism of Γ -modules:

$$L^2(\mathbb{H}^d, \Omega^p \otimes \mathbb{C}^r) \simeq L^2(\Gamma) \otimes L^2(\mathbb{H}^d, \Omega^p). \quad (33)$$

Here Γ acts on $L^2(\Gamma)$ by the left regular representaiton and we regard $L^2(\mathbb{H}^d, \Omega^p)$ is the trivial module. Since Hodge Laplacian Δ^p on $L^2(\mathbb{H}^d, \Omega^p \otimes \mathbb{C}^r)$ commutes with Γ so does $e^{-t\Delta^p}$. Let U be the fundamental domain of Γ and ψ its characteristic function. Using (14) von Neumann trace of $e^{-t\Delta^p}$ is given by ([1])

$$\tau(e^{-t\Delta^p}) = \text{Tr}(\psi \cdot e^{-t\Delta^p} \cdot \psi),$$

which is equal to $I_p(t)$. Let us put

$$\zeta_2(s) = \sum_p (-1)^p p \cdot \zeta_2^{(p)}(s), \quad \zeta_2^{(p)}(s) = \frac{1}{\Gamma(s)} M(\tau(e^{-t\Delta^p}))(s).$$

Then *the analytic L^2 -torsion* of (X, ρ) is defined to be

$$\tau_{an}^{(2)}(X, \rho) = \exp(-\frac{1}{2}\zeta_2'(0)).$$

[9]**Lemma 6.4** (see also **Appendix, Lemma 5.1** below) and the computation in §2.2 show that $M(I_p)(0)$ is a rational multiple of $\text{vol}(X)/\pi$ and that

$$\frac{d}{ds} \zeta_2^{(p)}(s)|_{s=0} = M(I_p)(0).$$

Now **Theorem 2.2** implies

Theorem 4.1. *Let h be the order of $R_X(\rho, z)$ at the origin. Then there is a rational number α such that*

$$\lim_{z \rightarrow 0} \left\{ \frac{d}{dz} \log R_X(\rho, z) - \frac{h}{z} \right\} - 2 \sum_{j=0}^n (-1)^j \delta(X, \rho) = \alpha \cdot \log \tau_{an}^{(2)}(X, \rho).$$

For example suppose $d = 3$. Since ([9] **Corollary 6.7**)

$$\log \tau_{an}^{(2)}(X, \rho) = M(I_0)(0) = \frac{r}{6\pi} \text{vol}(X),$$

we obtain

$$\lim_{z \rightarrow 0} \left\{ \frac{d}{dz} \log R_X(\rho, z) - \frac{2h^1(X, \rho)}{z} \right\} = -18 \cdot \log \tau_{an}^{(2)}(X, \rho).$$

5 Appendix

We will show **Fact 3.1** under the assumption that ρ is cuspidal and $d = 3$. In the following the Mellin transform of a function f will be denoted by $M(f)$:

$$M(f)(s) = \int_0^\infty f(t)t^{s-1}dt.$$

Lemma 5.1. *Let t be a positive number.*

1.

$$\int_{-\infty}^\infty e^{-t\lambda^2} d\lambda = \sqrt{\pi}t^{-\frac{1}{2}}.$$

2. For a positive integer k ,

$$\int_{-\infty}^\infty e^{-t\lambda^2} \lambda^{2k} d\lambda = \frac{\sqrt{\pi}(2k-1)!!}{2^k} t^{-\frac{1}{2}-k}.$$

3. Let c be a positive number and P an even polynomial. Then the Mellin transform of $\int_{-\infty}^\infty e^{-t(\lambda^2+c^2)} P(\lambda) d\lambda$ is meromorphically continued to \mathbb{C} . It is regular at $s = 0$ and

$$M\left(\int_{-\infty}^\infty e^{-t(\lambda^2+c^2)} P(\lambda) d\lambda\right)(0) = -2\pi \int_0^c P(iy) dy.$$

Proof. See [5] **Lemma 3**.

□

We put

$$\theta_p(t) = \text{Tr}[e^{-\Delta_x^p}] - h^p(X, \rho).$$

Since $H^0(X, \rho)$ vanishes Selberg trace formula shows

$$\theta_0(t) = \delta_0(t) = h_0(t) + e_0(t), \tag{34}$$

where

$$\begin{aligned} e_0(t) &= i_0(t) + u_0(t) \\ &= \frac{r \cdot \text{vol}(X)}{4\pi^2} \int_{-\infty}^\infty e^{-t(\lambda^2+1)} \lambda^2 d\lambda + \frac{\delta(X, \rho)}{2\pi} \int_{-\infty}^\infty e^{-t(\lambda^2+1)} d\lambda \\ &= \frac{r \cdot \text{vol}(X)}{8\pi\sqrt{\pi}} e^{-t} t^{-\frac{3}{2}} + \frac{\delta(X, \rho)}{2\sqrt{\pi}} e^{-t} t^{-\frac{1}{2}}. \end{aligned}$$

By **Lemma 5.1**, $M(e_0)(s)$ is regular at the origin and

$$M(e_0)(0) = \frac{r}{6\pi} \text{vol}(X) - \delta(X, \rho).$$

If $t > 0$ is sufficiently small,

$$h_0(t) \sim \frac{a}{\sqrt{4\pi t}} e^{-\frac{c_X^2}{4t}},$$

where c_X is the length of minimal closed geodesic. Since $\theta_0(t)$ exponentially decays as $a \rightarrow \infty$ so does $h_0(t)$. Therefore $M(h_0)(s)$ is an entire function and $M(\theta_0)(s)$ is a meromorphic function on the whole plane regular at the origin. Notice that $\Gamma(s)$ has simple pole with residue 1 at $s = 0$.

Proposition 5.1. $\zeta_X^{(0)}(s, \rho) = M(\theta_0)(s)/\Gamma(s)$ satisfies the following properties.

1. It is a meromorphic function on \mathbb{C} and vanishes at the origin.

2.

$$\frac{d}{ds} \zeta_X^{(0)}(s, \rho)|_{s=0} = M(\theta_0)(0) = M(h_0)(0) + \frac{r}{6\pi} \text{vol}(X) - \delta(X, \rho).$$

By definition the functional determinant is

$$\det \Delta_X^p = \exp\left(-\frac{d}{ds} \zeta_X^{(p)}(s, \rho)|_{s=0}\right).$$

Hence

$$-\log \det \Delta_X^0 = M(\theta_0)(0) = M(h_0)(0) + \frac{r}{6\pi} \text{vol}(X) - \delta(X, \rho). \quad (35)$$

We put

$$\eta_1(t) = h_1(t) + e_1(t) - h^1(X, \rho),$$

where

$$\begin{aligned} e_1(t) &= i_1(t) + u_1(t) \\ &= \frac{r \cdot \text{vol}(X)}{\pi^2} \int_{-\infty}^{\infty} e^{-t\lambda^2} (\lambda^2 + 1) d\lambda + \frac{\delta(X, \rho)}{\pi} \int_{-\infty}^{\infty} e^{-t\lambda^2} d\lambda \\ &= \frac{r \cdot \text{vol}(X)}{2\pi\sqrt{\pi}} (2t^{-\frac{1}{2}} + t^{-\frac{3}{2}}) + \frac{\delta(X, \rho)}{\sqrt{\pi}} t^{-\frac{1}{2}}. \end{aligned}$$

Then

$$\theta_1(t) = \eta_1(t) + \theta_0(t).$$

In order to investigate the Mellin transform of η_1 we consider

$$\mu_1(t) = \eta_1(t) + (h^1(X, \rho) - e_1(t)) \cdot \chi_{(0,1]} \quad (36)$$

$$= h_1(t) - (h^1(X, \rho) - e_1(t)) \cdot \chi_{(1,\infty)}, \quad (37)$$

where χ is a characteristic function. For a sufficiently small positive number t , (37) shows

$$\mu_1(t) \sim h_1(t) \sim \frac{a}{\sqrt{4\pi t}} e^{-\frac{c_X^2}{4t}}$$

and (36) implies for sufficiently large t

$$\mu_1(t) \sim \eta_1(t) \sim e^{-\gamma t}, \quad \gamma > 0.$$

Thus $M(\mu_1)(s)$ is an entire function. If $\text{Res} > \frac{3}{2}$,

$$\begin{aligned} M(\eta_1)(s) &= M(\mu_1)(s) + \int_0^1 (e_1(t) - h^1(X, \rho)) t^{s-1} dt \\ &= M(\mu_1)(s) - \frac{h^1(X, \rho)}{s} + \left(\frac{r \cdot \text{vol}(X)}{\pi \sqrt{\pi}} + \frac{\delta(X, \rho)}{\sqrt{\pi}} \right) \frac{1}{s-1/2} + \frac{r \cdot \text{vol}(X)}{2\pi \sqrt{\pi}} \frac{1}{s-3/2}. \end{aligned}$$

and if $\text{Res} < \frac{1}{2}$,

$$\begin{aligned} M(h_1)(s) &= M(\mu_1)(s) - \int_1^\infty (e_1(t) - h^1(X, \rho)) t^{s-1} dt \\ &= M(\mu_1)(s) - \frac{h^1(X, \rho)}{s} + \left(\frac{r \cdot \text{vol}(X)}{\pi \sqrt{\pi}} + \frac{\delta(X, \rho)}{\sqrt{\pi}} \right) \frac{1}{s-1/2} + \frac{r \cdot \text{vol}(X)}{2\pi \sqrt{\pi}} \frac{1}{s-3/2}, \end{aligned}$$

we see that both $M(\eta_1)(s)$ and $M(h_1)(s)$ are meromorphically continued to the whole plane as the same function. Moreover we find that $M(\eta_1)(s) + h^1(X, \rho)/s$ is regular at the origin. Together with **Proposition 5.1** this shows

Proposition 5.2. $\zeta_X^{(1)}(s, \rho) = M(\theta_1)(s)/\Gamma(s)$ satisfies the following properties.

1. It is a meromorphic function on \mathbb{C} and is regular at the origin. Moreover

$$\zeta_X^{(1)}(0, \rho) = -h^1(X, \rho).$$

2.

$$\frac{d}{ds} \zeta_X^{(1)}(s, \rho)|_{s=0} = M(\theta_0)(0) + \lim_{s \rightarrow 0} \{\Gamma(s) h^1(X, \rho) + M(\eta_1)(s)\}.$$

By **Proposition 2.1**, **Corollary 2.1**, **Proposition 2.4** and (34) we obtain

$$\begin{aligned} s_0(z+1) &= L(e^t h_0)(z) \\ &= L(e^t \delta_0)(z) - L(e^t i_0)(z) - L(e^t u_0)(z) \\ &= L(e^t \delta_0)(z) + \frac{r \cdot \text{vol}(X)}{2\pi} z^2 - \delta(X, \rho). \end{aligned}$$

By **Lemma 2.6** $L(e^t \delta_0)(z)$ is an odd function and thus

$$s_0(1-z) + s_0(1+z) = \frac{r}{\pi} \text{vol}(X) z^2 - 2\delta(X, \rho).$$

Since $s_j(z)$ is the logarithmic derivative of $S_j(z)$, (35) yields

$$\begin{aligned} \log S_0(2) - \log S_0(0) &= \int_0^1 (s_0(1+z) + s_0(1-z)) dz \\ &= \frac{r \cdot \text{vol}(X)}{3\pi} - 2\delta(X, \rho) \\ &= -2 \log \det \Delta_X^0 - 2M(h_0)(0). \end{aligned}$$

The equation ([5], pp535 (13)):

$$M(h_0)(0) = -\log S_0(2).$$

and (35) implies

Proposition 5.3.

$$\log(S_0(0)S_0(2)) = 2\log \det \Delta_X^0 = -2M(\theta_0)(0).$$

Now we will compute the Ray-Singer torsion. By **Proposition 5.1**, **Proposition 5.2** and **Proposition 5.3**

$$\begin{aligned} \zeta'_X(0, \rho) &= \frac{d}{ds} \zeta_X^{(1)}(s, \rho)|_{s=0} - 3 \frac{d}{ds} \zeta_X^{(0)}(s, \rho)|_{s=0} \\ &= \lim_{s \rightarrow 0} \{ \Gamma(s) h^1(X, \rho) + M(\eta_1)(s) \} + \log(S_0(0)S_0(2)). \end{aligned}$$

The arguments of pp.536 of [5](especially (25)) shows that the leading term of the Taylor expansion of $S_1(z+1)$ at the origin is $\delta z^{2h^1(X, \rho)}$. Here δ is given by

$$-\log \delta = \lim_{s \rightarrow 0} \{ \Gamma(s) h^1(X, \rho) + M(\eta_1)(s) \}.$$

Thus

$$\lim_{s \rightarrow 0} \{ \Gamma(s) h^1(X, \rho) + M(\eta_1)(s) \} = -\log(\lim_{z \rightarrow 0} z^{-2h^1(X, \rho)} S_1(z+1)),$$

and **Fact 2.1** shows

Theorem 5.1.

$$\lim_{z \rightarrow 0} z^{-2h^1(X, \rho)} R_X(z, \rho) = \exp(-\zeta'_X(0, \rho)).$$

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